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Stochastic Dynamics:
A Review of Stochastic Calculus of Variations

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We present the main results of a variational calculus for Markovian stochastic processes which allows us to characterize the dynamics of probabilistic systems by extremal properties for some functionals of processes. They generalize, by construction, the main variational formulations of classical dynamics. This framework is used for the dynamical analysis of Nelson's stochastic mechanics, an approach to quantum mechanics in which the concept of trajectory for particles still makes sense. The semiclassical limit is formulated in terms of the second variation of the starting functional. We also use the proposed stochastic calculus of variations in the context of statistical mechanics of systems far from equilibrium, namely, to solve the Onsager-Machlup problem.

PROLOGUE

This articles is a review of a minimal probabilistic extension of the classical calculus of variations developed in Geneva and Princeton by K. Yasue and the author, and whose main motivation is the characterization of dynamics for nondeterministic systems. The emphasis of the review is on the applications of this variational point of view in theoretical physics, and the necessary fundamentals of probability theory are only briefly summarized in the first section. Indeed, our approach would be very hypothetical without the knowledge of some clear physical interpretations for the simplest probabilistic generalization of the classical dynamical laws. Such an interpretation exists here, since this minimal extension corresponds to stochastic mechanics, a new frame for quantum mechanics proposed by E. Nelson in 1966. The possibility of finding again stochastic mechanics in a variational context including Lagrangian and Hamiltonian stochastic versions of

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dynamical laws suggests that, contrary to a too frequent idea, this physical theory is not reduced to an accidental coincidence with the Schrödinger equation. Some physically completely independent problems are also investigated from the stochastic variational point of view.

1. INTRODUCTION

It seems evident today that the paradigm of classical dynamics with smooth and stable trajectories $t \mapsto X(t)$ is not sufficient for the description of all the natural phenomena. Many of them, even in a purely classical context, seem more compatible with a probabilistic frame in which the irregularity of the paths is easily included. An open problem is then: How to discover all the possible dynamics of such probabilistic systems?

The way illustrated here is a stochastic calculus of variations (Yasue, 1981a, b) such that the realized dynamics extremalize some "action" functional of the processes. We will work mainly with processes for which the principle of causality holds, namely, Markovian processes.

After a succinct summary on the stochastic processes considered here, viz., time-symmetrical continuous semimartingales (Section 2), we give the main results of the stochastic calculus of variations (Section 3). In particular, expressed for the deterministic processes, which are trivially Markovian, they restore all the principles of the variational formulation of classical dynamics (Goldstein, 1980; Arnold, 1976). The third section examines the minimal extension of classical mechanics in this stochastic dynamical frame, namely, Nelson's stochastic mechanics (Nelson, 1966, 1976, 1984a). To paraphrase Caratheodory (1965-1967), the point of view of this section is that "the (stochastic) calculus of variations should be the servant of (stochastic) mechanics." The variational approach suggests a larger dynamical frame for this theory, with Lagrangian and Hamiltonian formulations. Some new results are obtained, for example, the equation of motion for the first quantum correction to a classical trajectory, namely, the stochastic equation of Jacobi. It describes the semiclassical limit of quantum mechanics.

Section 5 is devoted to the investigation of the dynamics in nonequilibrium statistical thermodynamics. A very interesting approach to these problems was initiated a long time ago by Onsager and Machlup (1953). We will find a close connection between this approach and the inverse problem of stochastic calculus of variations. In fact, the Onsager-Machlup problem (Graham, 1978) can be formulated in the most general case as an inverse problem of stochastic calculus of variations. The story of the variational principles is long and fascinating. It is nicely summarized in the book of Yourgrau and Mandelstam (1979).

2. NELSON RANDOM PROCESSES

$(\Omega, \mathfrak{a}, P)$ is our basic probability space, where the sample space Ω is the set of possible events ω, the sample points, $\mathfrak{a}$ the sigma-algebra of the observable events, and P the probability measure. $X: (\Omega, \mathfrak{a}) \to (\mathbb{R}^n, \mathfrak{B}^n)$, where $\mathfrak{B}^n$ is a Borel sigma-algebra, is a random variable if it is measurable. With respect to an arbitrary given sub-sigma-algebra σ of $\mathfrak{a}$, the random variable X is no longer measurable in general. However, there exists a unique random variable Y in $L^1(\Omega, \sigma, P|_\sigma)$ coarser than X but with the same value, on the average, on any $B \in \sigma$. This is the conditional expectation of X under the condition σ, denoted by $Y \equiv E[X|\sigma]$. The conditional probability of an event A under the condition $\sigma \subset \mathfrak{a}$ is defined by

$$P(A|\sigma) = E[\chi_A|\sigma] \tag{1}$$

where χ_A is the characteristic function of A.

A random process $X_t(\omega) \equiv X(t, \omega)$ is a family of $\mathbb{R}^n$-valued random variables indexed by a time interval. For a fixed $\omega \in \Omega$, $t \to X_t(\omega)$ is a trajectory of the process; in the following, it will be always continuous. A filtration $\mathfrak{P}_t$ is a growing family of sub-sigma-algebras of $\mathfrak{a}$ in relation to which each X_t is adapted (that is measurable). So, the "natural" filtration $a \le t$ is simply generated by the past of the process $a \le t \equiv a\{X(s); s \le t\}$. Let us denote by $\mathfrak{a} \ge t$ and η_t, respectively, the sigmas-algebra of the future and of the present for X_t.

X is a Markov process if the past and the future are independent when the present is known, that is if for each time and all events $A \in \mathfrak{a} \le t$, $B \in \mathfrak{a} \ge t$ we have (Dellacherie and Meyer, 1975)

$$P(AB|\eta_t) = P(A|\eta_t) \cdot P(B|\eta_t) \tag{2}$$

or equivalently, if f and p are any random variables adapted to the past and the future, respectively,

$$E[p \cdot f|\eta_t] = E[p|\eta_t]E[f|\eta_t] \tag{2'}$$

Since this definition is symmetrical in time, if $X(t)$ is a Markov process, $\bar{X}(t) \equiv X(-t)$ is another one and the kinematics of these two processes will not be independent. Let us also observe that for the choice $\mathfrak{a} \le t = \mathfrak{a} \ge t = \eta_0$ at each time, the process is simply deterministic and the Markov property is a version of the causality principle.

If X_t is adapted for a filtration $\mathfrak{P}_t$, X_t is a $\mathfrak{P}_t$-martingale if

$$E[X_t|\mathfrak{P}_s] = X_s \tag{3}$$

It is well known that the Wiener process W_t is an $\mathfrak{a} \leq t$-martingale; on any time interval, its length is infinite. Some processes are fortunately less pathological: their trajectory is of bounded variation.

A $\mathfrak{P}_t$ (local) semimartingale X admits the decomposition

$$X_t = X_0 + B_t + M_t \tag{4}$$

with $B_0 = M_0 = 0$ and M_t is a continuous (local) $\mathfrak{P}_t$-martingale and B_t is a continuous $\mathfrak{P}_t$-adapted process of bounded variations. [The "localization" property requires in fact the use of a sequence of $\mathfrak{P}$ stopping times, cf. Itô (1978).] For X_t and $\mathfrak{P}_t$ given, (4) is unique and called the Meyer canonical decomposition.

Nelson considered (and we shall follow him) $\mathfrak{P}_t$ semimartingales X_t in $\mathbb{R}^n$ whose canonical decomposition is explicitly given by

$$X(t) = X(0) + \int_0^t DX(s)\, ds + \mathfrak{P}\!\!\int_0^t \sigma(s)\, dW(s) \tag{5}$$

where $W(s)$ is the Wiener process on $\mathbb{R}^n$ and the two following limits belong to $L^1(\Omega)$ and are continuous from $I = [0, T]$ in $L^1(\Omega)$

$$DX(t) = \lim_{h \downarrow 0} E\left[\frac{X(t+h) - X(t)}{h} \middle| \mathfrak{P}_t\right] \tag{6}$$

$$\sigma^2(t) = \lim_{h \downarrow 0} E\left[\frac{\{X(t+h) - X(t)\}^2}{h} \middle| \mathfrak{P}_t\right] \tag{7}$$

The first integral in (5) is a Stieltjes integral for each sample and the second one a classical (Itô's) $\mathfrak{P}_t$-martingale integral. In particular, $X(t)$ is a $\mathfrak{P}_t$-martingale iff $DX(t) = 0$, $t \in I$ (Nelson, 1976). Let us denote, with Itô (1978), by $\mathcal{Q}_{\mathfrak{P}}$ the family of all continuous (local) $\mathfrak{P}_t$ semimartingales which contains X and by $\mathcal{L}_{\mathfrak{P}}(dX)$ the family of (progressively) measurable processes Y such that

$$Y(\cdot, \omega)\sigma(\cdot, \omega) \in L^2[0, t] \qquad \text{a.s.}$$

and

$$Y(\cdot, \omega) DX(\cdot, \omega) \in L^1[0, t] \qquad \text{a.s.}$$

For such a $Y \in \mathcal{L}_{\mathfrak{P}}(dX)$, the $\mathfrak{P}$-stochastic integral of Y based on dX is defined as the $\mathfrak{P}_t$ semimartingale

$$\mathfrak{P}\!\!\int_0^t Y \cdot dX \equiv \int_0^t Y(s) DX(s)\, ds + \mathfrak{P}\!\!\int_0^t Y(s)\sigma(s)\, dW(s) \tag{8}$$

Since the last integral is a $\mathfrak{P}_t$ martingale the (absolute) expectation yields

$$E\left[\oint_0^t Y\,dX\right]=E\left[\int_0^t Y(s)DX(s)\,ds\right] \tag{9}$$

This $\mathfrak{P}$-stochastic integral is not symmetrical in time. If $dZ = Y \cdot dX$ denotes a process Z_t in $\mathcal{Q}_{\mathfrak{P}}$ such that

$$Z_t - Z_0 = \oint_0^t Y\,dX$$

we can consider an n-dimensional process $Z=(Z_1,\ldots,Z_n)$ with $dZ_i = Y_i \cdot dX_i$. Then the process $F=f(Z)$, for $f\in C^2(\mathbb{R}^n\to\mathbb{R})$ satisfies the following chain rule:

$$dF=\sum_i \partial_i f(Z)\,dZ_i+\tfrac{1}{2}\sum_{i,j}\partial_i\partial_j f(Z)\,dZ_i\,dZ_j \tag{10}$$

In particular, for $f(Z_1,Z_2)=Z_1Z_2$, one obtains the important integration by parts formula

$$d(Z_1Z_2)=Z_2\,dZ_1+Z_1\,dZ_2+dZ_1\,dZ_2 \tag{11}$$

whose extra term $dZ_1\,dZ_2$, the quadratic variation of Z_1 and Z_2, is a clue of this time asymmetry.

A decreasing filtration $\mathfrak{F}_t$ is a time-reversed filtration if $\bar{\mathfrak{F}}_t \equiv \mathfrak{F}_{-t}$ is a growing filtration as before. X is a continuous (local) $\mathfrak{F}_t$ semimartingale, $X\in\mathcal{Q}_{\mathfrak{F}}$, if $\bar{X}(t)=X(-t)$ is an $\bar{\mathfrak{F}}_t$ semimartingale. $Y\in\mathcal{L}_{\mathfrak{F}}(dX)$ if $\bar{Y}(t)=Y(-t)\in\mathcal{L}_{\bar{\mathfrak{F}}}(d\bar{X})$. Then one defines the $\mathfrak{F}$-stochastic integral of Y based on dX by

$$\oint_0^t Y\,dX \equiv \oint_{-t}^0 \bar{Y}\,d\bar{X} \tag{12}$$

so that, if the $\bar{\mathfrak{F}}_t$ canonical decomposition $\bar{X}_t$ is

$$\bar{X}(t)=\bar{X}(0)+\int_0^t D\bar{X}(s)\,ds+\oint_0^t \bar{\sigma}(s)\,d\bar{W}_*(s) \tag{5'}$$

Equation (12) means

$$\oint_0^t Y\,dX=\int_{-t}^0 \bar{Y}D\bar{X}\,ds+\oint_{-t}^0 \bar{Y}\bar{\sigma}\,d\bar{W}_*(s)$$

Now, using the definitions (6) and (7), one verifies that $D\bar{X}(s) = -D_*X(-s)$ and $\bar{\sigma}(s) = \sigma_*(-s)$, where

$$D_*X(t) = \lim_{h\downarrow 0} E\left[\frac{X(t)-X(t-h)}{h}\middle|\mathfrak{F}_t\right] \tag{6'}$$

$$\sigma_*^2(t) = \lim_{h\downarrow 0} E\left[\frac{\{X(t)-X(t-h)\}^2}{h}\middle|\mathfrak{F}_t\right] \tag{7'}$$

and then the $\mathfrak{F}$-stochastic integral of Y based on dX is the $\mathfrak{F}_t$ semimartingale

$$\mathfrak{F}\!\!\int_0^t Y\cdot dX = \int_t^0 Y(s)D_*X(s)\,ds + \mathfrak{F}\!\!\int_t^0 Y(s)\sigma_*(s)\,dW_*(s) \tag{8'}$$

It follows from the martingale property of the last integral that

$$E\left[\mathfrak{F}\!\!\int_0^t Y\cdot dX\right] = E\left[\int_t^0 Y(s)D_*X(s)\,ds\right] \tag{9'}$$

The properties of this time-reversed integral are analogous to the properties of the $\mathfrak{P}$ integral. In particular, it is not symmetrical in time. For this reason, Itô (1978) introduced the two-sided symmetric stochastic integral of Y based on dX and relative to $(\mathfrak{P}, \mathfrak{F})$ for $X \in \mathcal{Q}_{\mathfrak{P}} \cap \mathcal{Q}_{\mathfrak{F}}$ and $Y \in \mathcal{L}_{\mathfrak{P}}(dX) \cap \mathcal{L}_{\mathfrak{F}}(dX)$ by

$$\int_s^u Y_0\,dX \equiv \frac{1}{2}\left[\int_s^u Y\,dX + \int_u^s Y\,dX\right], \qquad s \le t \le u \tag{13}$$

Using equations (9) and (9′), it is clear that

$$E\left[\int_s^u Y_0\,dX\right] = E\left[\int_s^u Y(t)\frac{1}{2}[DX(t) + D_*X(t)]\,dt\right] \tag{13'}$$

The main advantage of this symmetric integral is that the chain rule takes the same form as in classical calculus. Namely, in the same conditions as for (10), but with $f \in C^3(\mathbb{R}^n \to \mathbb{R})$

$$dF = \sum_i \partial_i f(Z) \circ dZ_i \tag{14}$$

In particular, for $f(Z_1, Z_2) = Z_1 Z_2$

$$d(Z_1Z_2) = Z_2 \circ dZ_1 + Z_1 \circ dZ_2 \tag{15}$$

It is also useful to introduce two other notions of symmetric integral. First of all, and without reference to the filtrations, one defines the symmetric Stratonovich integral by the limit in probability

$$\int_s^u Y \circ dX = \operatorname*{l.i.p.}_{|\Delta|\to 0} \sum_{i=1}^n \tfrac{1}{2}(Y_{t_i} + Y_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) \tag{16}$$

for Δ an arbitrary partition of $[s, u]$ and $|\Delta| \equiv \max|t_i - t_{i-1}|$, if the right-hand side of (16) exists. For example, if X and $Y \in \mathcal{Q}_{\mathfrak{P}}$ this is the case, and we obtain the forward Stratonovich integral (Itô, 1978),

$$\int_s^u Y \circ dX = \oint_s^u Y\,dX + \frac{1}{2}\int_s^u dY\,dX \tag{17}$$

In the same way, for X and $Y \in \mathcal{Q}_{\mathfrak{F}}$, we get the backward Stratonovich integral

$$\int_s^u Y \circ dX = \oint_s^u Y\,dX - \frac{1}{2}\int_s^u dY\,dX \tag{18}$$

If X and Y belong to $\mathcal{Q}_{\mathfrak{P}} \cap \mathcal{Q}_{\mathfrak{F}}$ we can write again equation (17) for X and Y interchanged and add the result to equation (18). Using the relation (16) we find the integration by parts formula

$$\int_s^u d(XY) = \oint_s^u Y\,dX + \oint_u^s X\,dY \tag{19}$$

or, after expectation,

$$E\left[X(t)Y(t)\Big|_s^u\right] = E\left[\int_s^u \{DX(t)Y(t) + X(t)D_* Y(t)\}\,dt\right] \tag{19'}$$

Now, let us assume that the base process X_t is a Markov process whose $\mathfrak{P}$ canonical decomposition (5) reduces to the integral form of the ($\mathfrak{P}$) stochastic (Itô's) differential equation

$$dX(t) = b(X(t), t)\,dt + \sigma(X(t), t)\,dW(t) \tag{20}$$

$X(0) = X_0(\omega) \in L^2(\Omega)$ [independent of the Wiener process on $\mathbb{R}^n$, $W(t)$], where b is a smooth $\mathbb{R}^n$-valued function, the forward drift, and σ an $n \times n$ real-valued function, the diffusion coefficient. Since by hypothesis $X(t)$ is also a time reversed semimartingale, it satisfies an ($\mathfrak{F}$) stochastic differential equation

$$dX(t) = b_*(X(t), t)\,dt + \sigma_*(X(t), t)\,dW_*(t) \tag{20'}$$

Using the general properties mentioned above, one can easily obtain the kinematical relations between the terms of equations (20) and (21).

It follows from the expectation of (17) that

$$E\left[\int_s^u Y\tfrac{1}{2}(DX - D_*X)\,dt\right] = -\frac{1}{2}E\left[\int_s^u dY\,dX\right]$$

For $X(t)$ generated by (20), any $Y = Y(X(t), t)$ is also in $\mathcal{Q}_{\mathfrak{P}}$ and, by Itô's calculus (Itô, 1975a),

$$dY_j\,dX_i = C_{ij}\frac{\partial Y_j}{\partial X_i}\,dt$$

where $C(X, t) \equiv \sigma \cdot \sigma^t(X, t)$, for $\cdot^t$ the transposed matrix. After integration by parts in the right-hand expectation, we have

$$\int_{\mathbb{R}} \int_s^u Y_j \tfrac{1}{2}(b_i - b_{*i})\rho \, dX \, dt = \int_{\mathbb{R}} \int_s^u \frac{1}{2} Y_j \frac{\partial}{\partial X_i}(C_{ij}\rho) \, dX \, dt$$

where ρ is the density of probability of the process X_t (we suppose that it exists) with respect to the Lebesgue measure. It follows from the validity of this last relation for any Y that

$$\frac{1}{2}(b - b_*) = \frac{1}{2\rho}\nabla(C\rho) \tag{21}$$

Taking into account the expectation of (18) we also find that, if $C_*(x, t) \equiv \sigma_* \sigma_*^t(X, t)$,

$$C_* = C \tag{22}$$

Before concluding this section, we quote a couple of useful formulas. For f and g two smooth functions in $C_0(\mathbb{R}^{n+1})$ and $X(t)$ generated by (20), (20′)

$$Df(X(t), t) = \left(\frac{\partial}{\partial t} + b_i \frac{\partial}{\partial x_i} + \frac{1}{2} C_{ij} \frac{\partial^2}{\partial X_i \, \partial X_j}\right) f(x(t), t) \tag{23}$$

and

$$D_* g(X(t), t) = \left(\frac{\partial}{\partial t} + b_{*i} \frac{\partial}{\partial X_i} - \frac{1}{2} C_{ij} \frac{\partial^2}{\partial X_i \, \partial X_j}\right) g(x(t), t) \tag{23′}$$

In the mathematical literature, the operators A and A_* defined by $D = \partial/\partial t + A$ and $D_* = \partial/\partial t + A_*$ are called forward and backward generators of the diffusion X_t. We will simply call this class of time symmetric continuous semimartingales "Nelson processes."

For a very pleasant general review on the theory of semimartingales, the reader may consult Williams (1981). A number of equations in this review hold only almost everywhere, but it will be clear from the context.

3. STOCHASTIC CALCULUS OF VARIATIONS

3.1. Stochastic Euler–Lagrange Equation

The goal of the stochastic calculus of variations is to obtain a minimal extension of the classical calculus of variations used in classical dynamics, particularly for Nelson processes. By "minimal" one means that in the limit of smooth trajectories ($D = D_* = d/dt$), any stochastic variational principle must be reduced to a classical one, in the form and in the result. Let $L \in C^2(\mathbb{R}^{3n} \times [t_a, t_b])$ a given real-valued and deterministic function and $X(t)$

a Nelson process on a time interval $I \supset [t_a, t_b]$. If the process $L(X(t), DX(t), D_*X(t), t)$ is integrable, one can introduce the real-valued action functional J associated to the Lagrangian L by

$$J: x \mapsto E\left[\int_{t_a}^{t_b} L(X(t), DX(t), D_*X(t), t)\, dt\right] \tag{24}$$

In this expression, D and D_* denote the conditional velocities (6) and (6′) but with respect to $\mathfrak{P}_t \cap \mathfrak{F}_t \equiv \eta_t$, namely the present at time t. The notation $D_{X_a}^{X_b}$ will be utilized for the set of Nelson processes with fixed end points (as random variables) $X(t_a) = X_a$, $X(t_b) = X_b$, and Δ for the set of Nelson processes $Z(t)$ with $Z(t_a) = Z(t_b) = 0$. An immediate question from the physical point of view is how to choose the form of the Lagrangian in (24). It will be possible to give a partial answer to this question in requiring some natural conditions for a probabilistic extension of classical dynamical systems (cf. Sections 3.3 and 3.6).

Let us recall that the functional $J = J[X]$ is differentiable in the sense of Frechet (or strongly) if one can write its increment as

$$\Delta J[\delta X] \equiv J[X + \delta X] - J[X] = \varphi(\delta X) + \theta(\|\delta X\|)$$

where the (first) "variation" φ is linear in δX. φ is called the Frechet derivative of J and is denoted by δJ.

The fundamental theorem of stochastic calculus of variations is due to Yasue (1981a):

Stochastic Hamilton Principle. A necessary and sufficient condition for $\bar{X} \in D_{X_a}^{X_b}$ to be a stationary point of the action functional J is that on $\bar{X}$

$$D\frac{\partial L}{\partial D_*\bar{X}(t)} + D_*\frac{\partial L}{\partial D\bar{X}(t)} - \frac{\partial L}{\partial \bar{X}(t)} = 0 \tag{25}$$

if this equation, interpreted as a generalization of the classical Euler–Lagrange equation, is well defined. To give a sense to (25), one considers that the stationary point $\bar{X}$ is embedded in a one parameter (ε) family of Nelson processes $X_t(\varepsilon) = X_t(0) + \varepsilon v(X_t(0), t)$, where $X_t(0) = \bar{X}_t$ has an a priori given quadratic variation. [This kinematical property is independent of the variational procedure. Notice that during this procedure, processes with other quadratic variations are used (cf. Itô's formula). From the physical point of view, they are virtual and without any connection with the equation of motion (25).] Also $v \in C^2$, $v(x, t_a) = v(x, t_b) = 0$. Using the Lagrange notation $\varepsilon v(\bar{X}(t), t) \equiv \delta X(t)$, $\bar{X}_t + \delta X_t$ is actually the variation of $\bar{X}_t$ in direction v. Then one computes the (first) variation of J on $\bar{X}$ in this direction by

$$\delta J[\bar{X}](\delta X)=E\left[\int_{t_a}^{t_b}\left\{\frac{\partial L}{\partial DX}\delta DX+\frac{\partial L}{\partial D_*X}\delta D_*X+\frac{\partial L}{\partial X}\delta X\right\}dt\right] \tag{26}$$

namely the Gateaux (or weak) derivative $(\partial/\partial\varepsilon)J[\bar{X}+\varepsilon v]|_{\varepsilon=0}$.

Using the integration by parts (19′), the commutation of the variation and the time derivatives, and the vanishing boundary values of $\delta X(t)$, equation (26) reduces to

$$0=E\left[\int_{t_a}^{t_b}\left(-D_*\frac{\partial L}{\partial DX}-D\frac{\partial L}{\partial D_*X}+\frac{\partial L}{\partial X}\right)\delta X\,dt\right] \tag{27}$$

by definition of a stationary point of J. Since (27) is true whatever $\delta X(t)$, one obtains (25). [Denote the random variable in parentheses by $Y(t)$ and choose $\delta X(t)=Y(t)f(t)$ for $f(t)$ such that $f(t_a)=f(t_b)=0$ (Nelson, 1984b).] Strictly speaking, one must precise the norm used on the space of Nelson processes. The usual L^2 norm $\|X\|^2=E[|X(t)|^2]$ is sufficient to obtain the stochastic Euler–Lagrange equation (25) but one also introduces other norms more appropriate for the variational problems.

On the (Banach) space of the Nelson processes such that $\sup_{t_a\le t\le t_b}E[|X(t)|^2]$ is finite, one defines the norm

$$\|X\|_0=\left(\sup_{t_a\le t\le t_b}E[|X(t)|^2]\right)^{1/2}$$

Since, in general, the action functional $J=J[x]$ will not be continuous with respect to $\|\cdot\|_0$, one also uses

$$\|X\|_1=\left(\sup_{t_a\le t\le t_b}\{E[|X(t)|^2]+E[|DX(t)|^2]+E[|D_*X(t)|^2]\}\right)^{1/2}$$

when the right-hand term exists. The associated topology is finer than for $\|\cdot\|_0$ since the norm $\|\cdot\|_1$ also controls the proximity of the velocities. In the conditions of the stochastic Hamilton principle, the action $J=J[X]$ is continuous with respect to $\|\cdot\|_1$. However, in the particular frame of Section 4, one easily constructs interesting "solvable" examples which are not in this second class of processes.

Other norms are proposed in Zheng and Meyer (1982/1983). Actually, the main results of this calculus of variations will be independent of the chosen particular norm.

In the classical limit of smooth trajectories, the given Lagrangian can also be interpreted as a function of three variables only,

$$L(X, DX=\dot{X}, D_*X=\dot{X}, t)\equiv L_c(X,\dot{X},t)$$

and the equation (25) will be simplified to the Euler–Lagrange equation. Before going further, let us give a more intuitive interpretation of the stochastic Hamilton principle via a notion of stochastic functional derivative (Zambrini, 1980a). The stochastic functional derivative $\delta J/\delta X(t)$ for the action J is defined by

$$\frac{\partial}{\partial \varepsilon} J[X+\varepsilon v]\bigg|_{\varepsilon=0} = E\left[\int_{t_a}^{t_b} \frac{\delta J}{\delta X(t)} \delta X(t)\, dt\right] \tag{28}$$

If the Nelson process $\delta X(t)$ is different from zero only in a h-neighborhood of t, the integral will disappear and the right-hand term of (28) will be close to

$$E\left[\frac{\delta J}{\delta X(t)} \Delta S\right] \tag{29}$$

for $\Delta S \sim \delta X(t) h$.

In using a polygonal approximation of $X(t, \omega)$ between (X_a, t_a) and (X_b, t_b) it is possible to verify that, as is clear from the definition (28),

$$\frac{\delta J}{\delta X(t)} = D\frac{\partial L}{\partial D_* X(t)} + D_* \frac{\partial L}{\partial DX(t)} - \frac{\partial L}{\partial X(t)} \tag{30}$$

In other words equation (25) expresses the fact that the stochastic functional derivative vanishes at every time inside the range t_a to t_b for a stationary point of J.

It will often be useful to know if a process is not only a stationary point of the action J but also a "local" (if one uses the chosen norm) extremal, that is a (local) minimum or maximum. It may be seen that if $\bar{X} = \bar{X}(t)$ is a local extremum of J in $D_{X_a}^{X_b}$, then it satisfies the stochastic Euler–Lagrange equation (25). The reciprocal proposition is, in general, not true.

Sufficient conditions for such an extremum are interesting since it also enables one to ignore the delicate question of existence of a minimum if one can exhibit the (unique) process which minimizes J, for example.

By definition, the action J is (strictly) convex on a domain D if for X and $X + \delta X$ two Nelson processes in D,

$$J[X+\delta X] - J[X] \geq \delta J[X](\delta X) \tag{31}$$

(with equality in X iff $\delta X = 0$). The Lagrangian L is (strongly) convex on $S \subset \mathbb{R}^{3n} \times \mathbb{R}$ if for all (x, y, z, t) and $(x+u, y+v, z+w, t)$ in S,

$$\begin{aligned} &L(x+u, y+v, z+w, t) - L(x, y, z, t) \\ &\quad \geq \partial_1 L(x, y, z, t)u + \partial_2 L(x, y, z, t)v + \partial_3 L(x, y, z, t)w \end{aligned} \tag{32}$$

[with equality in (x, y, z, t) only if $u = 0$ or $v = w = 0$].

Then we have the following theorem.

Convex Action Theorem. If $L(x, y, z, t)$ is (strongly) convex then

$$J[X]=E\left[\int_{t_a}^{t_b} L(X, DX, D_*X, t)\, dt\right]$$

is (strictly) convex on $D_{X_a}^{X_b}$, and there is a process X in $D_{X_a}^{X_b}$ such that

$$D\frac{\partial L}{\partial D_*X(t)}+D_*\frac{\partial L}{\partial DX(t)}-\frac{\partial L}{\partial X(t)}=0$$

on $[t_a, t_b]$ minimizes J on $D_{X_a}^{X_b}$ (and is unique).

Proof. By (strong) convexity of the Lagrangian,

$$E\left[\int_{t_a}^{t_b}\{L(X+\delta X, DX+D\delta X, D_*X+D_*\delta X, t)-L(X, DX, D_*X, t)\}\, dt\right]$$

$$\geq E\left[\int_{t_a}^{t_b}\left(\frac{\partial L}{\partial X}\delta X+\frac{\partial L}{\partial DX}D\delta X+\frac{\partial L}{\partial D_*X}D_*\delta X\right) dt\right]$$

then J is convex.

The equality is possible only if the integrand are equal (a.e.) that is, for L strongly convex, only if $\delta X\, D\, \delta X=\delta X\, D_*\, \delta X=0$. Therefore $\delta X(t)$ is a constant random variable, and since $\delta X(t_a)=0$ this constant is zero, hence J is strictly convex. Thanks to equations (26) and (27) the right-hand term of the inequality is also

$$\delta J[X](\delta X)=E\left[\int_{t_a}^{t_b}\left(\frac{\partial L}{\partial X}-D_*\frac{\partial L}{\partial DX}-D\frac{\partial L}{\partial D_*X}\right)\delta X\, dt\right]$$

then a solution $\bar{X}(t)$ of the stochastic Euler–Lagrange equation cancels δJ and thus $J[\bar{X}+\delta X]\geq J[\bar{X}]$ (with equality iff $\delta X=0$), that is, $\bar{X}$ minimizes J (and is unique). ■

A proof of the existence and unicity of the stationary point in the case of strictly convex action (and the main action functional used in this article (Section 4) will be strictly convex for a time interval $[t_a, t_b]$ small enough) was given by Zheng and Meyer (1982/1983).

3.2. Stochastic Hamilton Equations

Coming back to the Lagrangian $L=L(X, DX, D_*X, t)$, one defines generalized momenta by

$$\frac{1}{2}p=\frac{\partial L}{\partial DX},\qquad \frac{1}{2}p_*=\frac{\partial L}{\partial D_*X}\tag{33}$$

We will also suppose that L is strongly nondegenerate, and consequently that the two equations (33) can be solved in DX and D_*X, respectively. Then we have the following theorem.

Theorem. Stochastic Equations of Hamilton. The stochastic Euler–Lagrange equation (25) is equivalent to the system of stochastic Hamiltonian equations:

$$\frac{1}{2}DX = \frac{\partial H}{\partial p} \tag{34}$$

$$\frac{1}{2}D_*X = \frac{\partial H}{\partial p_*} \tag{35}$$

$$\frac{1}{2}(Dp_* + D_*p) = -\frac{\partial H}{\partial X} \tag{36}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{37}$$

where the Hamiltonian H is defined by

$$H = H(X, p, p_*, t) = \tfrac{1}{2}pDX + \tfrac{1}{2}p_*D_*X - L \tag{38}$$

Proof. By the symmetric chain rule (14) for the Nelson process $Z \equiv (X, p, p_*)$,

$$dH = \frac{\partial H}{\partial X} \circ dX + \frac{\partial H}{\partial p} \circ dp + \frac{\partial H}{\partial p_*} \circ dp_* + \frac{\partial H}{\partial t}\, dt \tag{39}$$

But we have also, using the definition (38),

$$dH = d(\tfrac{1}{2}pDX + \tfrac{1}{2}p * D_*X - L)$$

$$= \tfrac{1}{2}p \circ d(DX) + DX \circ \tfrac{1}{2}p + \tfrac{1}{2}p_* \circ d(D_*X) + D_*X \circ \tfrac{1}{2}\, dp_* - \frac{\partial L}{\partial X} \circ dX$$

$$- \frac{\partial L}{\partial DX} \circ d(DX) - \frac{\partial L}{\partial D_*X} \circ d(D_*X) - \frac{\partial L}{\partial t}\, dt$$

Taking into account the definition (33) of the momenta, the stochastic Euler–Lagrange equation (25) becomes $\partial L/\partial X = \frac{1}{2}Dp_* + \frac{1}{2}D_*p$. After insertion in the last expression of dH, this one reduces to

$$dH = DX \circ \tfrac{1}{2}\, dp + D_*X \circ \tfrac{1}{2}\, dp_* - \tfrac{1}{2}(Dp_* + D_*p) \circ dX - \frac{\partial L}{\partial t}\, dt$$

The comparison with equation (39) concludes the proof. Reciprocally, if the Hamiltonian variables satisfy the stochastic equations of Hamilton,

$X(t)$ verifies the stochastic Euler–Lagrange equation (25), then we have the equivalence. ■

Remarks. (1) It follows from equation (38) that $-\partial L/\partial X = \partial H/\partial X$.

(2) The "classical limit" of the definition (38) reduces to the classical notion of Hamiltonian, H_c.

(3) It is natural to call stochastic phase space the $3n$-space with coordinates $p_1 \cdots p_n; p_{*1} \cdots p_{*n}; x_1 \cdots x_n$. After the choice of an Hamiltonian formulation, the relations (33) are completely forgotten. The connection between the Hamiltonian variables and the time exists only by the equations of motion (34)–(36) (Goldstein, 1980).

An illustration of this aspect is the derivation of the stochastic equations of Hamilton directly from a variational principle.

Theorem. Least Action Principle in Phase Space. Let $Z(t) = (X(t), p(t), p_*(t)) \in D_{X_a, p_a, p_{*a}}^{X_b, p_b, p_{*b}}$ be a Nelson process in stochastic phase space. A necessary and sufficient condition for this process to be a stationary point of the functional

$$(X, p, p_*) \mapsto E\left[\int_{t_a}^{t_b} \{\tfrac{1}{2}pDx + \tfrac{1}{2}p_* D_* X - H(X, p, p_*, t)\}\, dt\right] \tag{40}$$

is that, on this process

$$\tfrac{1}{2}DX = \frac{\partial H}{\partial p}, \qquad \tfrac{1}{2}D_*X = \frac{\partial H}{\partial p_*}, \qquad \tfrac{1}{2}(Dp_* + D_*p) = -\frac{\partial H}{\partial X} \tag{41}$$

Proof. First observe that the integrand is nothing else than the Lagrangian L via the definition (38). By construction here, X, p, and p_* are therefore considered as independent variables of a new Lagrangian

$$\mathscr{L}(Z, DZ, D_*Z) = \mathscr{L}(X, p, p_*; DX, Dp, Dp_*; D_*X, D_*p, D_*p_*)$$

The stochastic Euler–Lagrange equations for p, p_*, and X give, respectively, the three equations (41).

The definition (38) of the Hamiltonian for the momenta (33) justifies calling an energy function the process

$$\varepsilon(X, DX, D_*X, t) = \frac{\partial L}{\partial DX} DX + \frac{\partial L}{\partial D_*X} D_*X - L \tag{42}$$

numerically identical to the Hamiltonian but a function of the Lagrangian variables.

3.3. Gauge Invariance

The Lagrangian, the energy, and the momenta are not defined univocally. Indeed, if $L = L(X, DX, D_*X, t)$ gives, by the stochastic Hamilton principle, an equation of motion (25), the same is true for the new Lagrangian

$$\tilde{L}(X, DX, D_*X, t) = L(X, DX, D_*X, t) - \tfrac{1}{2}(DF + D_*F)(X, t) \qquad (43)$$

for any smooth function $F = F(X, t)$.

The proof is immediate. If $J[X]$ denotes the original action (24), the new one associated to (43) is

$$\tilde{J}[X] = J[X] - E\left[\int_{t_a}^{t_b} \frac{1}{2}(DF + D_*F)(X, t)\, dt\right] \qquad (44)$$

that is, using equations (23) and (23′),

$$\tilde{J}[X] = J[X] - E\left[\int_{t_a}^{t_b} \left\{\frac{\partial F}{\partial t} + \frac{1}{2}(DX + D_*X)\partial_X F\right\} dt\right]$$

One directly verifies that the supplementary relation due to this new term in the stochastic Euler–Lagrange equation is an identity, and then the equation of motion is invariant under the transformation (43) designated by a gauge transformation. Two Lagrangians related by such a transformation are called equivalents. It also follows from the definitions (33) and (42) that

$$\tilde{p} = p - \partial_X F, \qquad \tilde{p}_* = p_* - \partial_X F \qquad (45)$$

and

$$\tilde{\varepsilon}(X, DX, D_*X, t) = \varepsilon(X, DX, D_*X, t) + \frac{\partial F}{\partial t} \qquad (46)$$

In other words, the transformation of the momenta and of the energy has the same form as in classical dynamics. Here, also, these variables are dependent of gauge.

It may be seen from equation (44) that the two actions $\tilde{J}[X]$ and $J[X]$ differ only by

$$E[F(X(t_b), t_b) - F(X(t_a), t_a)] \qquad (47)$$

namely, a constant, since we consider the fixed end points problem for $X(t) \in D_{X_a}^{X_b}$. More generally, we have the following lemma.

Lemma. Let $D \subset \mathbb{R}^{n+1}$, $F = F(X, t)$, and $g = g(X, t)$ a C^1 vector field and a C^1 scalar field arbitrary but such that the line integral

$$I[X] = E\left[\int_{X[t_a, t_b]} F \circ dX + g\, dt\right]$$

depends only on the end points of $X[t_a, t_b]$ in D. Then F and g satisfy the closure conditions

$$\frac{\partial F}{\partial t}=\frac{\partial g}{\partial X} \quad \text{and} \quad \frac{\partial F_i}{\partial X_j}-\frac{\partial F_j}{\partial X_i}, \qquad i, j=1, \ldots, n \tag{48}$$

Proof. The (stochastic) differential 1-form integrand of the action I is

$$\sum_{i=1}^{n} \int_{t_a}^{t_b} F_i(X_t, t) \circ dX_i(t)+g(X_t, t)\, dt$$

After expectation, using equation (13′),

$$\sum_{i=1}^{n} E\left[\int_{t_a}^{t_b} F_i(x_t, t) \cdot \frac{1}{2}(DX_i+D_* X_i)\, dt\right]+E\left[\int_{t_a}^{t_b} g(X_t, t)\, dt\right]$$

By hypothesis, this action J is a constant on $D_{X_a}^{X_b}$; then for all convenient δX, $\delta J[X](\delta X)=0$. Therefore by the stochastic Euler–Lagrange equation for X_i,

$$\left(\frac{\partial F_i}{\partial t}-\frac{\partial g}{\partial X_i}\right)+\sum_{j=1}^{n}\left(\frac{\partial F_i}{\partial X_j}-\frac{\partial F_j}{\partial X_i}\right) \frac{1}{2}(DX_i+D_* X_i)=0, \qquad i=1, \ldots, n$$

Since these relations must be true for any end points (t_a, X_a) and (t_b, X_b) in D, F and g must satisfy the closure conditions (48).

Conversely, if the domain D is simply connected (that is without holes) the relations (48) imply the existence of a C^1 function $S=S(X, t)$ such that the differential 1-form is exact,

$$dS=F \circ dX+g\, dt$$

Since the requirement of gauge invariance is supposed to be fundamental for any dynamical theory, we can use it to obtain an indication on the form of the Lagrangians in stochastic calculus of variations. In the classical case, if the Lagrangian L_c is linear in the velocity,

$$L_c(X, \dot{X}, t)=F(X, t) \dot{X}+g(X, t) \tag{49}$$

the corresponding classical Euler–Lagrange equation reduces to the closure conditions (48), which implies (in a domain without a hole) that the Lagrangian is the total derivative of a function $S=S(X, t)$,

$$\begin{aligned} dS &= F\, dX+g\, dt \\ &= L_c\, dt \end{aligned} \tag{50}$$

There is only one possible stochastic generalization of this situation, namely, as shown by the lemma,

$$dS = F \circ dX + g\,dt$$
$$\equiv L\,dt \tag{50'}$$

Since the expectation of equation (50′) gives

$$E[S(X(t_b), t_b) - S(X(t_a), t_a)]$$
$$= E\left[\int_{t_a}^{t_b} \left\{F \cdot \frac{1}{2}(DX + D_*X) + g\right\} dt\right]$$
$$\equiv E\left[\int_{t_a}^{t_b} L(X, DX, D_*X, t)\,dt\right] \tag{51}$$

it follows from the comparison between (49) and (51) that, in this particular case,

$$L(X, DX, D_*X, t) = \tfrac{1}{2}L_c(X, DX, t) + \tfrac{1}{2}L_c(X, D_*X, t) \tag{52}$$

This result will be useful for the third part of the review.

3.4. Constraints

Usually, in physics, the conditions of a realistic variational problem are not as simple as for the stochastic Hamilton principle. In particular, they involve different types of constraints. Here is a stochastic generalization of the most useful result in this direction. The problem is to find a necessary condition for $\bar{X}(t)$ in $D_{X_a}^{X_b}$ to be a local extremal of

$$J[X] = E\left[\int_{t_a}^{t_b} L(X, DX, D_*X, t)\,dt\right] \tag{53}$$

under the constraint during the variation

$$K[X] = E\left[\int_{t_a}^{t_b} M(X, DX, D_*X, t)\,dt\right] = \text{const} \tag{54}$$

We have the following theorem.

Stochastic Isoperimetric Theorem. If $\bar{X}(t) \in D_{X_a}^{X_b}$ is a local extremal point for this problem, one can find two constants μ and λ, not simultaneously zero, and such that, for $\mathcal{L} = \mu L + \lambda M$ and on $X = \bar{X}(t)$,

$$D_* \frac{\partial \mathcal{L}}{\partial DX} + D \frac{\partial \mathcal{L}}{\partial D_*X} - \frac{\partial \mathcal{L}}{\partial X} = 0 \tag{55}$$

Remark. One can obtain this result by an argument using the functional derivatives (Zambrini, 1980a). The following way is clearer.

Proof. By equation (26),

$$\delta J[\bar{X}](\delta X) = E\left[\int_{t_a}^{t_b} \left(\frac{\partial L}{\partial X}\delta X + \frac{\partial L}{\partial DX} D\delta X + \frac{\partial L}{\partial D_* X} D_*\delta X\right) dt\right]$$

and

$$\delta K[\bar{X}](\delta X) = E\left[\int_{t_a}^{t_b} \left(\frac{\partial M}{\partial X}\delta X + \frac{\partial M}{\partial DX} D\delta X + \frac{\partial M}{\partial D_* X} D_*\delta X\right) dt\right]$$

If $\delta K[\bar{X}](\delta X) = 0$ for all δX in Δ, the stochastic Euler–Lagrange equation for K,

$$D_* \frac{\partial M}{\partial DX} + D\frac{\partial M}{\partial D_* X} - \frac{\partial M}{\partial X} = 0$$

shows that $\mu = 0$ and $\lambda = 1$ are suitable for the conclusion. If $\delta K[\bar{X}](\delta X) \neq 0$, then one can find δY in Δ with $\delta K[\bar{X}](\delta Y) \neq 0$. Let us define for two fixed directions δX, δY, the two parameter functions

$$j_L(\varepsilon_1, \varepsilon_2) \equiv J[\bar{X} + \varepsilon_1 \delta X + \varepsilon_2 \delta Y]$$

and

$$k_M(\varepsilon_1, \varepsilon_2) \equiv K[\bar{X} + \varepsilon_1 \delta X + \varepsilon_2 \delta Y]$$

It may be seen that

$$\frac{\partial j_L(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_1} = \delta J[\bar{X} + \varepsilon_1 \delta X + \varepsilon_2 \delta Y](\delta X)$$

and then

$$\frac{\partial j_L(0, 0)}{\partial \varepsilon_1} = \delta J[\bar{X}](\delta X)$$

Similarly

$$\frac{\partial j_L(0, 0)}{\partial \varepsilon_2} = \delta J[\bar{X}](\delta Y)$$

On the other hand, for any δX in Δ, the Jacobian condition

$$\frac{\partial(j_L, k_M)}{\partial(\varepsilon_1, \varepsilon_2)}(0, 0) \equiv \begin{vmatrix} \delta J[\bar{X}](\delta X) & \delta J[\bar{X}](\delta Y) \\ \delta K[\bar{X}](\delta X) & \delta K[\bar{X}](\delta Y) \end{vmatrix} = 0 \tag{56}$$

is satisfied. Indeed, let us assume that $\bar{X}$ is a local minimal. If (56) is not true, one knows by the inverse function theorem that $(\varepsilon_1, \varepsilon_2) \mapsto (j_L(\varepsilon_1, \varepsilon_2), k_M(\varepsilon_1, \varepsilon_2))$ maps a neighborhood of $(0, 0)$ into a neighborhood of $(j_L(0,0), k_M(0,0))$. Hence we can choose ε_1 and ε_2 such that $j_L(\varepsilon_1, \varepsilon_2) = J[\bar{X}] - \alpha$ for some $\alpha > 0$, and $k_M(\varepsilon_1, \varepsilon_2) = k_M(0,0) = K[\bar{X}] = \text{const}$. But that is impossible since $\bar{X}$ is a local minimal.

Given that (56) is true, we obtain

$$\delta J[\bar{X}](\delta X) = \frac{\delta J[\bar{X}](\delta Y)}{\delta K[\bar{X}](\delta Y)}\, \delta K[\bar{X}](\delta X)$$

and then the constant

$$\mu = 1, \qquad \lambda = -\frac{\delta J[\bar{X}](\delta Y)}{\delta K[\bar{X}](\delta Y)} \tag{57}$$

are convenient for the conclusion because

$$\mu \delta J[\bar{X}](\delta X) + \lambda \delta K[\bar{X}](\delta X) = 0 \qquad \text{for any } \delta X \text{ in } \Delta \tag{58}$$

or

$$E\left[\int_{t_a}^{t_b} \left\{\left(\mu \frac{\partial L}{\partial X} + \lambda \frac{\partial M}{\partial X}\right) \delta X + \left(\mu \frac{\partial L}{\partial DX} + \lambda \frac{\partial M}{\partial DX}\right) D\delta X \right.\right.$$
$$\left.\left. + \left(\mu \frac{\partial L}{\partial D_* X} + \lambda \frac{\partial M}{\partial D_* X}\right) D_* \delta X \right\} dt\right]$$

The use of the integration by parts formula (19′) concludes the proof. ■

From the practical point of view, one interprets (58) as the variation $\delta(J + \lambda K) = 0$, which means that one can consider the modified Lagrangian $L + \lambda M$ without constraint. The constant λ will be called a Lagrange multiplier. The stochastic isoperimetric theorem is extended without difficulty to a finite number of constraints.

3.5. Transversal Conditions and Hamilton–Jacobi Equation

Some useful constraints are not of the preceding type, namely, they affect only the boundary conditions of the extremals for the given action. They are called transversal conditions.

Assume that we wish to minimize the action of *two* variables:

$$J = J[X, t] = E\left[\int_{t_a}^{t} L(X(s), DX(s), D_* X(s), s)\, ds\right] \tag{59}$$

on

$$D^N_{X_a} \equiv \{\text{Nelson processes } X(s);\ X(t_a) = X_a \text{ and } E[N(X(t), t)] = 0\} \tag{59'}$$

for $N: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ a given C^1 function such that $(\partial_X N, \partial_t N) \neq 0$. Notice that in order to define the variation of this action, one may introduce a norm on an appropriate product space, for example, $\|(X, t)\| = \|X\|_1 + |t|$.

If X is a local minimum, it satisfies the stochastic Euler-Lagrange equation on $[t_a, t]$ since the class of variations considered here contains Δ. Now the variation of J in the directions $(\delta X, \delta t)$ is by definition

$$\delta J[X, t](\delta X, \delta t) \equiv \frac{\partial}{\partial \varepsilon} J[X + \varepsilon \delta X, t + \varepsilon \delta t] \Big|_{\varepsilon = 0}$$

$$= \lim_{\varepsilon \to 0} \left\{ E\left[\int_{t_a}^{t+\varepsilon\delta t} \frac{\partial}{\partial \varepsilon} L[X + \varepsilon \delta X, s]\, ds \right] + E[L[X + \varepsilon \delta X, t + \varepsilon \delta t] \delta t] \right\}$$

$$= E\left[\int_{t_a}^{t} \delta L(s)\, ds \right] + E[L[X, t]\, \delta t] \tag{60}$$

where we use the condensed notations $L[X, s]$ for the integrand of (59) and $\delta L(s)$ for the integrand of (26). Given that $X(s)$ is a stationary point on $[t_a, t]$, we have

$$\delta J[X, t](\delta X, \delta t) = E\left[\left(\frac{\partial L}{\partial DX(s)} + \frac{\partial L}{\partial D_* X(s)} \right) \delta X(s) \Big|_{t_a}^{t} \right] + E[L[x, t]\, \delta t] \tag{61}$$

By hypothesis, the right-hand terms corresponds to the set of level 0 of the functional $K[X, t] \equiv E[N(X(t), t)]$. It may be found that

$$\delta K[X, t](\delta X, \delta t) \equiv \frac{d}{dt} E[N(X(t), t)] \delta t + E[\partial_X N\, \delta X] \tag{62}$$

Indeed, up to an irrelevant constant, $K[X, t]$ is also, by equations (44) and (47),

$$K[X, t] = E\left[\int_{t_a}^{t} \left\{ \partial_t N + \frac{1}{2}(DX + D_* X) \partial_X N \right\} ds \right]$$

It follows from the definition of the variation (60) that

$$\delta K[X, t](\delta X, \delta t) = E\left[\int_{t_a}^{t} \left(D_* \frac{\partial \mathscr{L}}{\partial DX} + D \frac{\partial \mathscr{L}}{\partial D_* X} - \frac{\partial \mathscr{L}}{\partial X} \right) \delta X\, dt \right] + E\left[\left(\frac{\partial \mathscr{L}}{\partial DX} + \frac{\partial \mathscr{L}}{\partial D_* X} \right) \delta X(s) \Big|_{t_a}^{t} \right] + E[\mathscr{L}[X, t] \delta t]$$

for $\mathscr{L}$ the "Lagrangian" $\frac{1}{2}(DN+D_*N)(X,t)$ of $K[X,t]$. Now, the stochastic Euler-Lagrange is identically zero for this Lagrangian and we get finally, since $\delta X(t_a)=0$,

$$\begin{aligned}\delta K[X,t](\delta X,\delta t)=E[\{\partial_X N\tfrac{1}{2}(DX+D_*X)+\partial_t N\}\delta t]\\ +E[\partial_X N\,\delta X]\end{aligned}$$

This is another form of (62).

Since δX and δt are independent, and by the hypothesis on N, this function of δX and δt is nonzero, then it follows from the stochastic isoperimetric theorem that there is a constant λ such that $\delta(J+\lambda K)=0$. The choices of variation for which first $\delta X(t_a)=\delta X(t)=0$ and then $\delta X(t_a)=\delta t=0$ give the two relations

$$E[L[X,t]]+\lambda\frac{d}{dt}E[N(X(t),t)]=0 \tag{63}$$

and

$$\left(\frac{\partial L}{\partial DX(t)}+\frac{\partial L}{\partial D_*X(t)}\right)+\lambda\partial_X N(X(t),t)=0 \tag{63$'$}$$

The elimination of λ between (63) and (63′) yields the sought transversality condition,

$$\begin{aligned}&\partial_X N(X(t),t)E[L[X,t]]\\ &\quad=\left(\frac{\partial L}{\partial DX(t)}+\frac{\partial L}{\partial D_*X(t)}\right)\frac{d}{dt}E[N(X(t),t)]\end{aligned} \tag{64}$$

If the initial point must also satisfy a constraint of the same type, $E[M(X(t_a),t_a)]=0$, a local extremal will be a solution of the stochastic Euler-Lagrange equation with the boundary condition (64) and

$$\begin{aligned}&\partial_X M(X(t_a),t_a)E[L[X_a,t_a]]\\ &\quad=\left(\frac{\partial L}{\partial DX(t_a)}+\frac{\partial L}{\partial D_*X(t_a)}\right)\frac{d}{dt_a}E[M(X(t_a),t_a)]\end{aligned} \tag{64$'$}$$

Any mixed boundary conditions are evidently also possible. Let us consider two examples:

(1) If $N(x,t)=t_b-t$, the condition (64) reduces to

$$\frac{\partial L}{\partial DX(t_b)}+\frac{\partial L}{\partial D_*X(t_b)}=0 \tag{65}$$

It is the "variable end point problem" in which the random variable $X(t_b)=X(t)$ is free.

(2) For the given Lagrangian L, let us introduce a function $S_L = S_L(X(t), t)$ by

$$N(X(t), t) \equiv S_L(X(t), t) - E\left[S_L(X(t), t) - S_L(X(t_a), t_a) - \int_{t_a}^{t} L[X, S]\, ds\right] \qquad (66)$$

Then $D^N_{X_a}$ defined in (59′) takes the form

$$\left\{\begin{array}{l}\text{Nelson processes}\\ X(s);\, X(t_a) = X_a \text{ and } E[S_L(X_a, t_a)] + E\left[\int_{t_a}^{t} L[X, s]\, ds\right] = 0\end{array}\right\} \qquad (67)$$

We will call the *wave front in t from the point X_a* the set of levels defined in this way.

Taking into account the particular expression (66), and if we assume that $E[L[X, t]] \neq 0$, the transversality condition (64) is simplified to

$$\partial_X S_L(X(t), t) = \frac{\partial L}{\partial DX(t)} + \frac{\partial L}{\partial D_* X(t)} \qquad (68)$$

This was essentially the choice of Yasue's (1981b) original paper. In the classical limit of deterministic trajectories, the left-hand term of the equality in (67) is nothing but the solution of the classical Hamilton-Jacobi equation with initial condition $S_L(X_a, t_a)$ (Arnold, 1976) and (68) the gradient condition $\partial_X S_{L_c} = p_c$ ($\cdot_c$ for "classical"). Now, classically, the Hamilton-Jacobi equation is fundamentally associated to the closure of a differential 1-form, namely,

$$dS_{L_c} = p_c\, dX - H_c\, dt \qquad (69)$$

Because of the presence of two distinct notions of velocities in the stochastic frame, we have no a priori reason to think that (69) is directly generalized for the stochastic Hamiltonian defined in (38) or even that the classical relation

$$S_{L_c}(x, t) - S_{L_c}(X_a, t_a) = \int_{X_a, t_a}^{X, t} L_c(X, \dot{X}, s)\, ds \qquad (70)$$

can always be extended to the process. But anyway, $S_L = S_L(X, t)$ is not completely defined by (68). If we denote by $\mathcal{M}(X(t), t)$ the right-hand vector field in (68) ($\mathcal{M}$ for "momentum"), we can introduce a scalar field $h = h(X(t), t)$ such that the line integral

$$E\left[\int_{X[t_a, t_b]} \mathcal{M}(X(t), t) \circ dX - h(X(t), t)\, dt\right] \qquad (71)$$

depends only on the end points. Moreover, there certainly exists a Lagrangian $\tilde{L}$ equivalent to the given one L (cf. Section 3.3) for which

$$E\left[\int_{X[t_a,t_b]} \mathcal{M}(X(t), t) \circ dX - h(X(t), t)\, dt\right] = E\left[\int_{t_a}^{t_b} \tilde{L}(X, DX, D_* X, t)\, dt\right] \tag{72}$$

Taking into account the result of Section 3.3 (for a simply connected domain) we know that there exists a C^1 function S such that

$$\partial_X S = \mathcal{M}(X(t), t) \tag{68'}$$

$$\partial_t S = -h(X(t), t) \tag{73}$$

Equation (73) will be interpreted as a stochastic version of the Hamilton–Jacobi equation.

We can observe, finally, that the stationarity requirement of the right-hand action in (72) restores the stochastic Euler–Lagrange equation for the original L (by gauge invariance), whereas the stationarity of the left-hand action implies the closure conditions associated to the integrated stochastic differential 1-form.

3.6. Maupertuis' Principle for Conservative Systems

In Section 3.3, we found that the only form of Lagrangian compatible with the requirement of gauge invariance is

$$L(X, DX, D_* X, t) = \tfrac{1}{2}L_c(X, DX, t) + \tfrac{1}{2}L_c(X, D_* X, t) \tag{52}$$

for L_c a classical Lagrangian linear in the velocity. It is then natural to suppose that some properties of the usual classical dynamical systems will survive for the stochastic dynamics. Let us introduce the class of Lagrangian of the form

$$L(X, DX, D_* X) = T(X, DX, D_* X) - V(X) \tag{74}$$

where $T \in C^2(\mathbb{R}^{3n})$ is a real-valued function homogeneous of degree 2 in the velocities DX and $D_* X$, called kinetic energy, and $V \in C^2(\mathbb{R}^{3n})$ a real-valued potential energy. It follows from the general definition (38) of the Hamiltonian (or from (42)) that the energy function associated to (74) is

$$\varepsilon(X, DX, D_* X) = T(X, DX, D_* X) + V(X) \tag{75}$$

For this stochastic version of conservative systems (Goldstein, 1980) we have the

Maupertuis Principle of Least Action (*Zambrini*, 1984). For the class of Nelson processes in $D_{X_a}^{X_b}$ such that

$$E[\varepsilon(X, DX, D_*X)] = h \qquad \text{(a constant)} \tag{76}$$

the stochastic Hamilton principle implies the stochastic Maupertuis principle, which means that if

$$\delta E\left[\int_{t_a}^{t_b} L(X, DX, D_*X)\, dt\right] = 0$$

then

$$\delta_{\text{th}} E\left[\int_{X[\cdot,\cdot]} 2T(X, DX, D_*X)\, dt\right] = 0 \tag{77}$$

where the index th in the right-hand variation denotes a variation which also modifies the interval of the time parameter for the processes satisfying the energy constraint (76).

Proof. First of all, let us examine the general relation between these two types of variations. According to (26) we can denote by δL the integrand of the Gateaux derivative, namely,

$$\delta L \equiv \frac{\partial L}{\partial X}\, \delta X + \frac{\partial L}{\partial DX}\, \delta DX + \frac{\partial L}{\partial D_* X}\, \delta D_* X \tag{78}$$

Now suppose that t itself is a deterministic differentiable function $t: [u_a, u_b] \to \mathbb{R}$ such that $dt/du = \phi > 0$. Under the bijective time change $T_\phi : X(u) \mapsto X(t(u)) \equiv \bar{X}(u)$ the two velocities (6) and (6′) are modified to

$$D_u \bar{X} = \phi Dx, \qquad D_{*u} \bar{X} = \phi D_* X \tag{79}$$

which yields for a variation δ_t in which the parameter is also varied

$$\begin{aligned} \delta_t DX &= \delta_t\left(\frac{D_u \bar{X}}{\phi}\right) \\ &= D\delta X - DX \frac{d}{dt}(\delta_t t) \end{aligned} \tag{80}$$

where $\delta_t t$ is a variation of the parameter. Similarly,

$$\delta_t D_* X = D_* \delta X - D_* X \frac{d}{dt}(\delta_t t) \tag{80'}$$

It may be seen that the relation between (78) and this new one is

$$\delta_t L = \delta L - \left(DX \frac{\partial L}{\partial DX} + D_* X \frac{\partial L}{\partial D_* X} \right) \frac{d}{dt}(\delta_t t) \tag{81}$$

For example, by definition of the potential and kinetic energies,

$$\delta_t V = \delta V$$

$$\delta_t T = \delta T - 2T \frac{d}{dt}(\delta_t t)$$

which implies that the stochastic Hamilton principle can also be formulated as

$$E\left[\int_{t_a}^{t_b} \left\{ \delta_t T + 2T \frac{d}{dt}(\delta_t t) - \delta_t V \right\} dt \right] = 0 \tag{82}$$

Now the sense of the constraint (76) in this frame is that one must modify the time interval in order to maintain the energy constant; then

$$\delta_t E[T + V] = E[\delta_t T + \delta_t V] = 0$$

and (82) reduces to

$$2E\left[\int_{t_a}^{t_b} \{\delta_t T\, dt + T\, d(\delta_t t)\} \right] = 0$$

Since $d(\delta_t t) = \delta_t \phi\, du$ we finally get the conclusion

$$E\left[\int_{u_a}^{u_b} 2\delta_t(T\phi)\, du \right] = \delta_t E\left[\int_{t_a}^{t_b} 2T\, dt \right] = 0$$

Notice that by homogeneity of the kinetic energy and the definitions of the momenta (23), $2T = \frac{1}{2}(pDX + p_* D_* X)$, which implies [cf. equations (9) and (9′)] that Maupertuis' principle also means that

$$\delta_{\text{th}} \frac{1}{2} E\left[\oint p\, dX + \oint p_*\, dX \right] = 0 \tag{83}$$

This formulation is again in accordance with our initial principle of minimal extension of the classical variational principles. Actually, it is possible to prove more concerning the Maupertuis principle: it is in fact equivalent to Hamilton's principle for the considered stochastic conservative systems (Zambrini, 1984). This last reference explores the main variational principles associated with the variation δ_t.

4. STOCHASTIC MECHANICS

4.1. Equations of Motion

In this section, we shall systematically use the dynamical information provided by the stochastic calculus of variations for the minimal extension of classical mechanics. It was noted by Goldstein (1980) that for almost all problems of interest in classical mechanics, the Lagrangian L_c has the form

$$L_c(X, \dot{X}, t) = T(X, \dot{X}, t) + L_1(X, \dot{X}, t) - L_0(X, t) \tag{84}$$

where T, L_1, and L_0 are, respectively, homogeneous functions of degree 2, 1, and 0 in the velocity. A typical realization of (84) is

$$L_c(X, \dot{X}, t) = \tfrac{1}{2}\dot{X}_i G_{ij} \dot{X}_j + \dot{X}_i A - V \tag{84'}$$

where $G = (G_{ij})(X, t)$, $G_{ij} = G_{ji}$, $A = (A_i)(X, t)$, $i, j = 1$ to n. This type of Lagrangian describes, for example, the dynamics of a (system of) charged particles in any electromagnetic field and other potential V.

Now the requirement of gauge invariance (Section 3.3) and also the class of Lagrangians introduced in Section 3.6 for Maupertuis' principle suggests the following minimal extension:

$$L(X, DX, D_*X, t) = \tfrac{1}{2}L_c(X, DX, t) + \tfrac{1}{2}L_c(X, D_*X, t) \tag{85}$$

where L_c is of the form (84').

Before exploring the dynamical consequences of this choice, let us observe that, as in classical mechanics, for a time interval $[t_a, t_b]$ small enough, the kinetic term dominates and the associated action functional is strictly convex (cf. the convex action theorem, Section 3.1). For a more precise statement, consult Zheng and Meyer (1982/1983). We consider the simplest case where $G = IM$ ($I = n \times n$ identity matrix, M = positive constant) and $A = 0$, namely, if $|\cdot|$ is the Euclidean norm,

$$L(X, DX, D_*X, t) = \frac{M}{4}|DX|^2 + \frac{M}{4}|D_*X|^2 - V(X, t) \tag{86}$$

or equivalently, according to the definition (38) of the Hamiltonian

$$\begin{aligned} H(X, p, p_*, t) &= \tfrac{1}{2}H_c(X, p, t) + \tfrac{1}{2}H_c(X, p_*, t) \\ &= \frac{1}{4M}|p|^2 + \frac{1}{4M}|p_*|^2 + V(X, t) \end{aligned} \tag{87}$$

since the momenta (33) are reduced to

$$p = MDX, \qquad p_* = MD_*X \tag{88}$$

In the limit of smooth trajectories, L, H, p, and p_* are clearly simplified to the classical definitions for a particle of mass M.

It follows from the stochastic equations of Hamilton [equations (34)-(36)] that the dynamics of the system with the Hamiltonian (87) is described by

$$DX = \frac{p}{M} \tag{89}$$

$$D_* X = \frac{p^*}{M} \tag{90}$$

$$\frac{1}{2}(Dp_* + D_* p) = -\frac{\partial V}{\partial X} \tag{91}$$

This set of equations is equivalent to

$$\frac{M}{2}(DD_* X + D_* DX) = -\frac{\partial V}{\partial X} \tag{92}$$

which is nothing but the stochastic Euler-Lagrange equation (25) for the Lagrangian (86). It is natural to interpret equation (92) as the stochastic version of Newton's equation.

Using the method presented in Section 3.5 on transversal conditions, we can construct the wave front in t from the point X_a for the given Lagrangian. Then, by equation (68), one can find a function $S_L = S_L(X(t), t)$ such that

$$\frac{\partial S_L}{\partial X}(X(t), t) = \frac{M}{2}(DX(t) + D_* X(t)) \tag{93}$$

Now if we assume that the base process $X(t)$ is a Markov process whose $\mathfrak{P}$ (respectively, $\mathfrak{F}$) decomposition reduces to an Itô stochastic differential equations (20) and (20′), an important kinematical characteristic of the stationary point for the starting action must be precised. Recall that, by construction, the quadratic variation of this process embedded in a one-parameter family is given a priori. In our case, we impose, using the notation (22),

$$C(X(t), t) = \lim_{\Delta t \downarrow 0} E\left[\frac{\{X(t+\Delta t) - X(t)\}^2}{\Delta t}\middle| X(t)\right] = \frac{\hbar}{M} I \tag{94}$$

where $I = n \times n$ is the identity matrix, and $\hbar$ is Planck's constant.

Such a kinematical constraint was first proposed by Feynman in his original space-time approach to nonrelativistic quantum mechanics (Nelson, 1979; Albeverio and Hoegh-Krohn, 1974). With this assumption, the sought stochastic process is now completely characterized. According to Section 3.5, one can always choose the function S of (93) in such a way that

$$E\left[\int_{X[t_a,t_b]} Mv(X(t), t)\circ dX - h(X(t), t)\, dt\right] = E[S(X_b, t_b) - S(X_a, t_a)] \tag{95}$$

namely, the left-hand line integral is independent of the path between $X(t_a) = X_a$ and $X(t_b) = X_b$.

We have introduced the notation for (93) [cf. equations (20), (20′)]

$$v(X(t), t) \equiv \tfrac{1}{2}(DX(t) + D_* X(t)) = \tfrac{1}{2}[b(x(t), t) + b_*(x(t), t)] \tag{96}$$

In other words, since the form of $v = v(X(t), t)$ is already imposed by the transversal condition (93), $h = h(X(t), t)$ in (95) is adapted so as to close the stochastic differential 1-form integrated in the action (95).

With such a function S, one can construct $\psi_t \in L^2(\mathbb{R}^n)$ by

$$\psi(X, t) \equiv \rho^{1/2}(X, t)\, e^{(i/\hbar)S(X,t)} \tag{97}$$

where ρ is the density of probability of $X(t)$ (with respect to the Lebesgue measure d^nX) and verify that ψ satisfies

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\Delta\psi + V\psi \tag{98}$$

which is the Schrödinger equation for the (system of) particles of mass M in the potential V of the starting classical Lagrangian used in (85).

In this frame, namely, Nelson's stochastic mechanics (Nelson, 1966, 1976, 1984a), the Born interpretation ceases being an interpretation and becomes a fact:

$$|\psi(x, t)|^2\, d^nX = P(X(t) \in d^nX) \tag{99}$$

Therefore, the minimal extension of classical mechanics, in this stochastic variational frame, corresponds to quantum mechanics. Observe that all the hypotheses used to obtain this representation of the Schrödinger equation are of a very mechanical nature, except one of them, namely, the kinematical constraint (94). For the reader worried (as the author sometimes is) about the existence of two distinct velocities in this interpretation, it will be interesting to translate the constraint (94) into the Hamiltonian language:

Lemma 1. The kinematical constraint (94) implies the "canonical commutation relation"

$$E[p_j(t)X_i(t) - X_i(t)p_{*j}(t)] = -\delta_{ij}\hbar \tag{100}$$

Proof. In our simplest case, $p = MDX$ and $p_* = MD_*X$ by (88), that is, according to equations (20) and (20′), $p = Mb(x, t)$ and $p_* = Mb_*(x, t)$. The left-hand side of (100) is modified to $ME[X(b - b_*)]$. Using (94) in the general relation (21) we get

$$b - b_* = \frac{\hbar}{M} \frac{\nabla \rho}{\rho} \tag{101}$$

and then the conclusion after integration by parts.

The sense of the "classical limit of smooth trajectories" becomes clearer thanks to (100). The constant $\hbar$ is a measure of the difference between the two momenta p and p_*. There is no way to avoid this dual description of the velocities in a stochastic approach of quantum mechanics.

Let us indicate a useful consequence of this commutation relation.

Lemma 2. In stochastic mechanics, the following relation is true for each $t \in [t_a, t_b]$:

$$\tfrac{1}{2}[DDX(t) - D_*D_*X(t)] = 0 \tag{102}$$

Proof. By the integration by parts formula (19′) and (100),

$$E\left[\int_{t_a}^{t_b} \frac{1}{2}[DDX(t) - D_*D_*X(t)]X(t)\, dt\right] = E[[DX(t) - D_*X(t)]X(t)|_{t_a}^{t_b}]$$
$$= 0$$

Since the starting action must be zero for any solution $X(t)$ of the stochastic Newton equation (92) and for any time interval $[t_a, t_b] \subset I$, we get the result.

The identity (102) takes a more familiar form if one introduces, following Nelson (1966), the osmotic velocity

$$u(X(t), t) \equiv \tfrac{1}{2}[b(X(t), t) - b_*(X(t), t)] \tag{103}$$

Indeed, a straightforward computation using the definitions (96) and (103) shows that (102) reduces to

$$\frac{\partial u}{\partial t} = -\frac{\hbar}{2M}\Delta v - \text{grad}\, v \cdot u \tag{104}$$

Taking into account the relation (101) between the osmotic velocity and the probability density ρ, this implies that ρ satisfies the equation of continuity

$$\frac{\partial \rho}{\partial t} = -\text{div}(v\rho) \tag{105}$$

Therefore, the kinematical character of this equation in stochastic mechanics follows from the commutation relation (100).

Before concluding this section, we give a brief discussion about the existence of the random processes involved in stochastic mechanics. By the construction (95) and the definition (101), we have

$$v = \nabla S, \qquad u = \tfrac{1}{2}\nabla \log \rho \qquad (\text{for } \hbar = M \equiv 1) \tag{106}$$

which means, by (97), that b and b_* can be directly expressed in terms of the wave function ψ as

$$b = \operatorname{Re}\frac{\nabla\psi}{\psi} + \operatorname{Im}\frac{\nabla\psi}{\psi} \tag{107}$$

$$b_* = \operatorname{Im}\frac{\nabla\psi}{\psi} - \operatorname{Re}\frac{\nabla\psi}{\psi} \tag{107'}$$

A long-standing and delicate mathematical problem was the following. For a given reasonable potential V and a given solution $\psi(X, t)$ of the Schrödinger equation (98) does there exist a Nelson process $X(t)$ which satisfies the Born condition (99) on $[t_a, t_b]$ and whose forward and backward drifts are given by (107) and (107′)? The trouble is that at the nodes of the wave function these drifts are very singular.

This problem is nicely solved by E. A. Carlen in his Princeton thesis.

Carlen's Theorem (1984). Such a Nelson process $X(t)$ exists for V in the very large class of Rellich potentials, if the initial kinetic energy of the system is finite, namely, in the notation of Section 4.1, if

$$E[T(DX, D_*X)](t_a) = E[\tfrac{1}{4}b^2(X, t_a) + \tfrac{1}{4}b_*^2(X, t_a)] < \infty$$

This finite energy hypothesis is natural from the variational point of view. It is in fact a necessary condition for giving a sense to the stochastic Hamilton principle.

Finally, we may observe that, even without the introduction of the dynamical variables p and p_*, namely, without the construction of the stochastic equations of Hamilton (89)-(91), it is of course possible to obtain some interesting properties on the momentum, using directly the information contained in the Schrödinger equation (98) (more precisely in the Fourier transform of ψ). Consult Shucker (1980) on this point.

4.2. Hamilton–Jacobi, Conservation of Energy and Stationary States

In order to obtain the particular explicit form of the Hamilton-Jacobi equation in stochastic mechanics, it is sufficient to use equations (106) and the Schrödinger equation. We find that equation (73) becomes

$$\begin{aligned} \partial_t S &= -h(x, t) \\ &= \tfrac{1}{2}(\nabla u + u^2 - v^2) - V \end{aligned} \tag{108}$$

It satisfies the minimal extension principle since in the limit of smooth trajectories the osmotic velocity u is zero [cf. (103)] and then the right-hand term of (108) reduces to $-(\frac{1}{2}v^2+V)$, namely, minus the classical energy.

Notice that an independent variational principle, inspired by the methods of stochastic control theory, and which enables one to obtain (108), was proposed recently by Guerra and Morato (1983).

This function S, the quantum mechanical phase, enables one to close the stochastic differential 1-form of the action (95) and then to give the following probabilistic representation for the involved solution (97) of the Schrödinger equation ($\hbar = M = 1$),

$$\begin{aligned}\psi(X,t) &= \rho^{1/2}(X,t)\\ &\quad\times\exp\left\{ iE_{X,t}\left[S(X_a,t_a) + \int_{t_a}^{t} v(X_s,s)\circ dX_s \right.\right.\\ &\quad\left.\left. - \int_{t_a}^{t} h(X_s,s)\,ds \right]\right\}\end{aligned} \tag{109}$$

where $E_{X,t}$ is the conditional expectation with the condition $X(t)=X$.

Since the right-hand side of (108) is a stochastic version of energy, let us consider now the question of the conservation of energy.

Actually, we already know the probabilistic form of this relation. According to the constraint (76) used in the Maupertuis principle, it is, for a time-independent potential,

$$E[\varepsilon(X, DX, D_*X)] = E\left[\frac{M}{4}|DX|^2 + \frac{M}{4}|D_*X|^2 + V(X)\right] = \text{const} \tag{110}$$

The comparison between the right-hand term of (108), $-h$, and ε shows that we have two different notions of energy. Now any operational notion of energy, in quantum mechanics, is actually an expectation. Let us compute

$$E[h] = E[\tfrac{1}{2}(v^2 - u^2 - \nabla u) + V]$$

Taking into account the definition (106), we observe that $-\frac{1}{2}E[\nabla u] = E[u^2]$ and so

$$\begin{aligned}E[h] &= E[\tfrac{1}{2}(v^2+u^2)+V]\\ &= E\left[\frac{M}{4}b^2 + \frac{M}{4}b_*^2 + V\right]\end{aligned}$$

by (86) and (103). This is precisely the right-hand side of (110). In other words, for any $t \in [t_a, t_b]$,

$$E[h(X(t),t)] = E[\varepsilon(X(t), b(X(t),t), b_*(x(t),t))] \tag{111}$$

The particular case $v=0$ (the "symmetric case," cf. Section 5.4) is very important from the physical point of view. According to the equation of continuity (105), it corresponds to a stationary situation (time independent) for a conservative system, $\partial\rho/\partial t=0$.

For example, if (97) is of the form

$$\psi(X,t)=\varphi_n(X)\,e^{-iE_nt} \tag{112}$$

where $\varphi_n\in L^2(\mathbb{R}^n)$ is an eigenfunction of the stationary Schrödinger equation associated to equation (98) for the energy eigenvalue E_n [of course on the assumption that the given potential $V=V(X)$ admits such a solution], the corresponding Nelson process is a strictly stationary Markov process whose invariant measure is $\rho d^nX=|\varphi_n(x)|^2d^nX$ and the constant in (110) is precisely the eigenvalue E_n. A stationary situation for which $v=\nabla S(X)\neq 0$ is also possible. It happens if (97) can be written as

$$\psi(X,t)=\rho(X)\,e^{-i[Et-S(X)]} \tag{113}$$

Such a situation is common for stationary states of the Hydrogen atom for ex. (cf. Section 4.3).

There is a lot of interesting information about the stationary states in stochastic mechanics. The mathematical construction of the associated Nelson processes was investigated in Albeverio and Hoegh-Kroh (1974) and Carmona (1979a). In particular, it is possible to analyze the effect of the nodes of ψ on the process $X(t)$, namely, the existence of impassable barriers (mathematically, the decomposition of the invariant measure in different time ergodic components). For a discussion of these questions, one can consult Nelson (1984a), Albeverio et al. (1984), and Nagasawa (1980).

4.3. Gauge Transformation

Coming back to the classical Lagrangian (84′) (with $G=IM$) for the more general case of a particle (mass M, charge unity) in an electromagnetic field with scalar and vector potential ϕ and A:

$$L_c(X,\dot X,t)=\frac{M}{2}|\dot X|^2+\dot XA(X,t)-\phi(X,t)-V(X,t) \tag{114}$$

where V describes now all other potentials which act on the particle. Using the same argument as in Section 4.1, we find that the associated Lagrangian of stochastic mechanics is

$$L(X,DX,D_*X,t)=\frac{M}{4}|DX|^2+\frac{M}{4}|D_*X|^2 +\frac{1}{2}(DX+D_*X)A-\phi-V \tag{115}$$

which means that the starting action functional takes the form

$$J[X]=E\left[\int_{t_a}^{t_b}\left(\frac{M}{4}|DX|^2+\frac{M}{4}|D_*X|^2-\phi-V\right)dt+\int_{t_a}^{t_b}A\circ dX\right] \quad (116)$$

The definition (38) gives the corresponding Hamiltonian

$$H(X,p,p_*,t)=\tfrac{1}{2}H_c(X,p,t)+\tfrac{1}{2}H_c(X,p_*,t)$$
$$=\frac{1}{4M}|p-A|^2+\frac{1}{4M}|p_*-A|^2+\phi+V \quad (117)$$

which yields the stochastic equations of Hamilton

$$DX=\frac{p-A}{M} \quad (118)$$

$$D_*X=\frac{p_*-A}{M} \quad (119)$$

$$\frac{1}{2}(DP_*+D_*p)=\frac{1}{2M}[(p-A)+(p_*-A)]\partial_X A-\partial_X\phi-\partial_X V \quad (120)$$

After substitution (120) modifies to

$$\frac{M}{2}(DD_*X+D_*DX)=-\partial_X[\phi+V-\tfrac{1}{2}(DX+D_*X)A]$$
$$-\tfrac{1}{2}(DA+D_*A) \quad (121)$$

Now, with the vector identity

$$\partial_X[\tfrac{1}{2}(DX+D_*X)A]-\tfrac{1}{2}(DA+D_*A)=-\partial_t A+\tfrac{1}{2}(DX+D_*X)\wedge \operatorname{rot} A$$

the right-hand term of equation (121) becomes

$$F_L\equiv-\partial_X\phi-\partial_t A+\tfrac{1}{2}(DX+D_*X)\wedge\operatorname{rot} A-\partial_X V \quad (122)$$

It follows from the usual definitions of the field E and B,

$$E=-\partial_X\phi-\partial_t A \quad (123)$$

$$B=\operatorname{rot} A \quad (124)$$

that F_L defines the stochastic version of the Lorentz force plus the external one, as required,

$$F_L=E+\tfrac{1}{2}(DX+D_*X)\wedge B-\partial_X V \quad (125)$$

In this more general situation, the transversality condition (68) means that there exists $S_L=S_L(X(t),t)$ such that

$$\partial_X S_L(X,t)=\frac{M}{2}(DX+D_*X)+A\equiv p+p_* \quad (126)$$

As previously, the closure of the associated stochastic differential 1-form introduces a new function S which also satisfies (126). If ρ is the probability density of the underlying Nelson process $X(t)$ one can consider in $L^2(\mathbb{R}^n)$,

$$\psi(X, t) \equiv \rho^{1/2}(X, t)\, e^{(i/\hbar)S(X,t)} \tag{127}$$

and verify that the equation of motion with the Lorentz force (125),

$$\frac{M}{2}(DD_*X + D_*DX) = F_L$$

is equivalent to the Shrödinger equation for this physical situation (Nelson, 1966)

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2M}[-i\hbar\nabla - A]^2\psi + \phi\psi + V\psi \tag{128}$$

We make two remarks. The first one concerns the appearance, in the starting action functional (116), of the Fisk–Stratonovich integral for the vector potential. We may note that the necessity of making this choice is already known (in another frame, of course) for the conventional derivation of the Schrödinger equation (128) via the path integral methods (Schulman, 1981).

The second remark is related to the physical problem of gauge transformation in quantum mechanics. The difficulty of this question, in usual quantum mechanics and in comparison with classical mechanics, is mainly L. S., 1981). At this level, the problem of gauge invariance is nonexistent in stochastic mechanics.

For example, under the gauge transformation

$$\begin{aligned} A &\rightsquigarrow \tilde{A} = A - \partial_X F \\ \phi &\rightsquigarrow \tilde{\phi} = \phi + \partial_t F \end{aligned} \tag{129}$$

our Lagrangian (115) is transformed according to (43) into

$$L \rightsquigarrow \tilde{L} = L - \tfrac{1}{2}(DF + D_*F) \tag{130}$$

without effects on the stochastic Euler–Lagrange equation. Now, it follows from the transformation (45) of the momenta that the new transversality condition implies the existence of $S_{\tilde{L}}$ such that

$$\partial_X S_{\tilde{L}}(X, t) = \mathcal{M} - \partial_X F \tag{131}$$

for $\mathcal{M} \equiv \partial_X S_L$. If we denote, as in Section 3.5, by h the scalar field used to close the stochastic differential 1-form associated to L, namely, $-h = \partial_t S_L$ the transformed scalar field will be

$$\partial_t S_{\tilde{L}}(X, t) = -h - \partial_t F \tag{132}$$

The relations (131) and (132) mean that, up to an irrelevant constant (a "global gauge"), the two quantum mechanical phases are simply related by

$$S_{\tilde{L}}(X, t) = S_L(X, t) - F(X, t) \tag{133}$$

and then the wave function by

$$\tilde{\psi}(X, t) = \psi(X, t)\, e^{-iF(X,t)} \tag{134}$$

It is straightforward to verify that $\tilde{\psi}$ satisfies

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = \frac{1}{2M}[-i\hbar\nabla - \tilde{A}]^2 \tilde{\psi} + \tilde{\phi}\tilde{\psi} + V\tilde{\psi} \tag{135}$$

In Pauli's terminology, this is a "local gauge transformation of the first kind" (Pauli, 1941).

3.4. Semiclassical Limit

In the framework of stochastic mechanics, the semiclassical limit corresponds to comparing a classical, differentiable trajectory with a semiclassical one which trembles a little in its neighborhood, as depicted in Figure 1. Therefore, it is natural to envisage it as an expansion around a differentiable (that is classical) process $\bar{x}(t)$,

$$X(t) = \bar{x}(t) + \delta X(t) \tag{136}$$

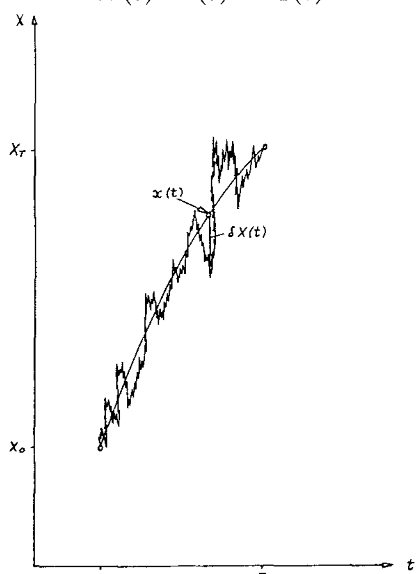

Fig. 1. The classical solution between two fixed points and a realization of the corresponding semiclassical process.

where $X(t)$ is a Nelson process of quadratic variation (94) and $\delta X(t)$ is another in Δ. One can assume that $\dot{\bar{x}}(t)$ is bounded on the time interval of interest, hence $\bar{x}(t)$ is of bounded variation. Since the quadratic variation of a semimartingale is unaffected by the addition of a process of bounded variation, $\delta X(t)$ has itself the same quantum quadratic variation as $X(t)$.

For the Lagrangian (86) of quantum mechanics, we assume that the action functional J is twice (Fréchet) differentiable and then its increment, computed along the classical solution $\bar{x}(t)$, is

$$\Delta J[\delta X] = \delta J[\bar{x}](\delta X) + \frac{1}{2!}\delta^2 J[\bar{x}](\delta X) + \theta(\|\delta X\|^2)$$

This expansion is convenient for describing the semiclassical limit. The first variation $\delta J[\bar{x}](\delta X)$ is given by equation (2.6), but here, since it is computed along the classical path $\bar{x}(t)$, the integration by parts yields a vanishing factor in the integrand, namely, the classical Newton equation for $\bar{x} = \bar{x}(t)$. By definition, the second variation $\delta^2 J$ involves the second-order term in the Taylor expansion

$$\frac{1}{2}E\left[\int_{t_a}^{t_b}\left\{\frac{\partial^2 L}{\partial X^2}(\delta X)^2 + 2\frac{\partial^2 L}{\partial X\,\partial DX}\,\delta X\,\delta DX + 2\frac{\partial^2 L}{\partial X\,\partial D_* X}\,\delta X\,\delta D_* X\right.\right.$$
$$\left.\left. + \frac{\partial^2 L}{\partial DX^2}(\delta DX)^2 + 2\frac{\partial^2 L}{\partial DX\partial D_* X}\,\delta DX\,\delta D_* X + 2\frac{\partial^2 L}{\partial D_* X^2}(\delta D_* X)^2\right\}dt\right]$$

namely, given that all the partial derivatives are computed on $(\bar{x}, \dot{\bar{x}}, \dot{\bar{x}}, t)$ and after integration by parts in the one-dimensional case (for simplicity we only consider this case in the following),

$$\frac{1}{2}E\left[\int_{t_a}^{t_b}\left\{-\frac{\partial^2 V}{\partial X^2}(\bar{x}(t))\,\delta X - \frac{M}{2}DD_*\delta X - \frac{M}{2}D_* D\delta X\right\}\delta X\,dt\right] \quad (137)$$

The requirement of stationarity gives us the equation of motion of the first quantum correction to the chosen classical solution

$$\frac{M}{2}(DD_* Z + D_* DZ) = -\frac{\partial^2 V}{\partial X^2}(\bar{x}(t))Z \quad (138)$$

One denotes now by Z the former process δX for underlining the fact that, as a solution of equation (138), the semiclassical process $Z(t)$ will be, of course, no more in Δ. In n dimensions, the right-hand term of (138) becomes $(-\partial^2 V/\partial X_i \partial X_j)Z_i$. According to the minimal extension principle, we call it the stochastic equation of Jacobi for the Nelson deviation process $Z(t)$. Notice that the right-hand term factor of Z is computed on the value $\bar{x}(t)$ of the classical solution at time t. It is evident that equation (138) can also be considered as the stochastic Euler–Lagrange equation for the following quadratic function in Z:

$$\frac{1}{2}\delta^2 J[\bar{x}]: Z \mapsto E\left[\int_{t_a}^{t_b}\left\{\frac{M}{4}DZ^2+\frac{M}{4}D_*Z^2-\frac{1}{2}\frac{\partial^2 V}{\partial X^2}(\bar{x})Z^2\right\}dt\right] \tag{139}$$

From our point of view, the stochastic equation of Jacobi (138) contains all the dynamical information on the quantum semiclassical limit (Zambrini and Yasue, 1982). It will be convenient to introduce the scalar product

$$(Z_1|Z_2) \equiv E\left[\int_{t_a}^{t_b} Z_1(t)Z_2(t)\,dt\right] \tag{140}$$

and then the norm

$$\|Z\|^2 = E\left[\int_{t_a}^{t_b} Z(t)^2\,dt\right]$$

for the sought Nelson process, and the stochastic version of differential operator

$$Y \equiv \frac{M}{2}(DD_* + D_*D) + \frac{\partial^2 V}{\partial X^2}(\bar{x}(t)) \tag{141}$$

in order to examine formally equation (138). Indeed, one verifies immediately, after integration by parts, that

$$\delta^2 J[\bar{x}](Z) = -(Z|YZ) - E\left[\frac{M}{2}Z(DZ+D_*Z)\Big|_{t_a}^{t_b}\right]$$

Now the domain D_Y of Y will be the set of Nelson processes such that the boundary term on the right-hand side is zero. In particular, a convenient choice corresponds to the following stochastic version of Neumann boundary conditions:

$$\tfrac{1}{2}(DZ+D_*Z)(t_a) = \tfrac{1}{2}(DZ+D_*Z)(t_b) = 0 \tag{142}$$

The following question is evidently interesting from the physical point of view of the semiclassical limit: How can we find a nontrivial Nelson process $Z(t)$ satisfying the Neumann conditions (142) and minimizing the quadratic functional $\delta^2 J[\bar{x}](Z)$ under the natural constraint

$$E\left[\int_{t_a}^{t_b} Z(t)^2\,dt\right] = 1? \tag{143}$$

It follows from the stochastic isoperimetric theorem (Section 3.4) that it must be a Lagrange multiplier μ such that the sought extremal satisfies

$$\frac{M}{2}(DD_*Z + D_*DZ) = -\frac{\partial^2 V}{\partial X^2}(\bar{x}(t))Z - \mu Z \tag{144}$$

in other words, using the notation (141),

$$YZ + \mu Z = 0 \tag{144'}$$

Observe that the number μ will be precisely the minimal value of the quadratic functional, since

$$0 = (Z|\, YZ + \mu Z) = -\delta^2 J[\bar{x}](Z) + \mu$$

Observe also that on the above-mentioned domain, the operator Y is symmetric, $(f|\,Yg) = (Yf|g)$ for any f and g in DY. The given problem is manifestly a stochastic version of a classical Sturm–Liouville problem in $L^2[t_a, t_b]$, namely, $Y_c Z + \mu Z = 0$ for the differential operator on the C^2 functions in $L^2[t_a, t_b]$ such that $\dot{Z}(t_a) = \dot{Z}(t_b) = 0$,

$$Y_c = M\frac{d^2}{dt^2} + \frac{\partial^2 V}{\partial X^2}(\bar{x}(t)) \tag{141'}$$

In the classical variational context, it is called the "Morse boundary value problem" (Morse, 1934). Y_c is a symmetric operator with a pure point spectrum. Its lowest eigenvalue μ_1 corresponds to the eigenfunction Z_1 minimizing the classical functional

$$\delta^2 J[\bar{x}](Z) = \int_{t_a}^{t_b} \left\{ \frac{M}{2}|\dot{Z}|^2 - \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(\bar{x}(t))Z^2 \right\} dt$$

under the constraint

$$\int_{t_a}^{t_b} Z(t)^2 \, dt = 1$$

The other eigenfunctions are only stationary points of this quadratic functional. A very simple but interesting limit case is the study of this problem on a short time interval $[t_a, t_b] \equiv [0, T]$. On this assumption, it may be seen that the term $(\partial^2 V/\partial X^2)(\bar{x}(t))Z$ is negligible, and therefore the same approximation yields the following equation for the semiclassical process:

$$\frac{M}{2}(DD_* Z + D_* DZ) = -\mu Z \tag{145}$$

with stochastic Neumann conditions.

As an illustration of the stochastic variational point of view, we shall solve (formally) the equation (145). We summarize the result in the following lemma.

Lemma. For any classical (differentiable) deviation $D(t)$ in $L^2[0, T]$ with Neumann boundary conditions $\dot{D}(0) = \dot{D}(T) = 0$, one can construct a gaussian Nelson process $Z(t)$, with stochastic Neumann condition, whose wave packet moves without spreading on $[0, T]$ and is centered on $D(t)$. Moreover, $Z(t)$ can be written as

$$Z(t) = \sum_{n=1}^{\infty} (Z|Z_n)Z_n(t) \tag{146}$$

where $\{Z_n(t)\}_{n\in\mathbb{N}}$ are orthonormal Nelson process solutions of the stochastic Morse boundary problem

$$\frac{M}{2}(DD_*Z_n + D_*DZ_n) = -\mu_n Z_n \tag{147}$$

Proof. An orthonormal basis of $L^2[0, T]$ which satisfies the classical Sturm Liouville problem $M\ddot{D}_n = -\mu_n D_n$ with classical Neumann conditions is given by $\{D_n(t)\}_{n\in\mathbb{N}} = \{(2/T)^{1/2} \cos \omega_n t\}_{n\in\mathbb{N}}$ for the positive eigenvalues $\mu_n = M\omega_n^2 = M(n\pi/T)^2$. Then any differentiable $D(t)$ can be written $D(t) = \sum_{n=1}^{\infty} (D, D_n)D_n(t)$, where $(\, , \,)$ is the scalar product in $L^2[0, T]$. Now let us consider a process $Z(t)$ solution of the stochastic differential equation on $[0, T]$

$$dZ(t) = \{\dot{D}(t) - [Z - D(t)]\}\, dt + (\hbar/M)^{1/2}\, dW(t) \tag{148}$$

for some initial condition in $L^2(\Omega)$. It is a Gauss-Markov process with the normal distribution $\mathcal{N}(D(t), \hbar/2M)$. In particular

$$\|Z\|^2 = \frac{T}{2} + \int_0^T D(t)^2\, dt$$

is finite. Using the relation (21) for $C = \hbar/M$, we get the backward drift $b_*(Z, t) = \dot{D}(t) + Z - D(t)$ and then

$$\tfrac{1}{2}[DZ(t) + D_*Z(t)] = \dot{D}(t) \tag{149}$$

which means that $Z(t)$ satisfies the stochastic Neumann condition (142) on $[0, T]$. Moreover, since the variance of $Z(t)$ is constant on this interval, we have no "spreading" of the associated wave packet.

Let us introduce an infinite collection of Nelson processes $\{Z_n(t)\}_{n\in\mathbb{N}}$ such that

$$dZ_n(t) = -\omega_n[Z - a_n(\cos \omega_n t - \sin \omega_n t)]\, dt + (\hbar/\gamma M)^{1/2}\, dW_n(t) \tag{150}$$

where the $W_n(t)$ are independent Wiener processes, ω_n as above

$$a_n \equiv \left(\frac{2}{T} \frac{\hbar}{M\omega_n}\right)^{1/2} \tag{151}$$

and γ a finite constant without dimension.

Note that (15.1) has a sense for T small enough. These processes are also Gauss–Markov processes and their normal distributions are

$$\mathcal{N}\left(a_n \cos \omega_n t, \frac{\hbar}{2M} \frac{1}{\gamma \omega_n} \right)$$

It follows from their independence that $(Z_n|Z_m) = \delta_{nm}$, and they also verify the stochastic Neumann conditions. A simple computation shows that each process $Z_n(t)$ satisfies the stochastic Morse boundary problem (142) for the above-mentioned eigenvalue μ_n.

On the other hand, if we denote $(Z|Z_n)$ by α_n, the processes $\alpha_n Z_n$ are also normal with distribution

$$\mathcal{N}\left(\alpha_n a_n \cos \omega_n t, \frac{\hbar}{2M} \frac{\alpha_n^2}{\gamma \omega_n} \right)$$

Now if, for each t, this series of processes converges in probability (or equivalently in mean square since they are Gaussian) the limit is also a Gaussian process, with distribution $\mathcal{N}(m(t), \sigma^2)$. By identification with the distribution of the Gauss Markov process $Z(t)$ and using the convergence of $D(t)$ in $L^2[0, T]$ we find compatible conditions on the coefficients $(D, D_n) \equiv \theta_n$ and α_n. The constant γ is determined by the constraint on the quantum mechanical quadratic variation of the limit process $Z(t)$.

Of course this construction is partially qualitative, but it convinces me that the stochastic equations of Jacobi (138) contains indeed all the physics of the semiclassical limit, and that the stochastic version (144) of the Morse boundary value problem deserves to be closely examined from the mathematical point of view. Instead of doing that, we prefer to comment on the physical sense of the preceding construction.

For each frequency ω_n, the process Z_n represents a mode of deviation from the classical path. The corresponding Gaussian wave packet moves without distortion. Up to the local scale of the fluctuations, it is a "coherent state" (Schrödinger, 1926; Glauber, 1963) of the oscillator ω_n, as it must be during a short interval of time.

In the flow of papers on the semiclassical limit, we may cite, as illustration of the conventional approach of quanum mechanics (Kac, 19xx; Hagedorn, 1981), for the connection with the nature of the underlying classical motion (Berry, 1981), and from the stochastic mechanics point of view but for stationary states (tunneling problems) (Jona-Lasinio et al., 1981a, b). For other aspects of Nelson's theory, in particular for the incorporation of spin into stochastic mechanics, the main source is, of course, Nelson's (1984a) book. Consult also Dankel (1971), Dohrn et al. (1979), Faris (1982), Guerra (1980), Dankel (1977).

In the framework of stochastic mechanics, the central motivation of this calculus of variation is to obtain new physical information both on the quantum kinematics and dynamics. From the kinematical point of view, for example, a quantum extension of the mechanical theory of constraints is now possible. From the dynamical point of view, the search for nonconservative quantum evolution (Yasue, 1978) will be simpler in this variational frame. More generally, for all the extensions of quantum mechanics in the domains where it is expected to be in probable contradiction with the phenomena, the variational point of view, close to the classical physical intuition, may enable us to guess more easily the new laws we are looking for. An example of this type of research is given by the impressive program initiated by Smolin (1983) on the relationship between quantum and gravitational phenomena.

5. ONSAGER–MACHLUP PROBLEM IN NONEQUILIBRIUM STATISTICAL MECHANICS

5.1. The Problem and its Various Approaches

Let us consider an open thermodynamical system which is not in equilibrium. Its state is characterized by a set of macroscopic variables $X(s)$ on $\mathbb{R}^n$ whose evolution is continuous in time. For numerous physically interesting situations (Enz, 1977), one may suppose that $X(s)$ is some Markoffian stochastic process. Such a framework begins with several trends in nonequilibrium thermodynamics. From the physical point of view, two main directions must be mentioned which both wish to determine (by different ways) some potentials, possibly dependent on time, from which the macroscopic properties of the system will be derived.

The first one is Prigogine's school (Nicolis and Prigogine, 1977), which is particularly interested in steady states far from equilibrium. The second one, initiated by Onsager and Machlup (1953), instead considers small departures from the equilibrium situation, modelized by "linear Langevin equations" for $X(s)$. This last approach was generalized by Graham (1978) to arbitrary stochastic differential equations in these variables of the form (20) but for homogeneous processes, namely,

$$dX(s) = b(X(s))\,ds + \sigma(X(s))\,dW(s) \tag{152}$$

The motivation of the one-dimensional (for simplicity) Onsager–Machlup problem is to find, for a given equation (152), a function $L_{OM} = L_{OM}(X, \dot{X})$ such that the following (formal) integral representation of the transition semigroup (Glimm and Jaffe, 1981) makes some sense:

$$(\text{Kernel } e^{tA})(x, y)$$
$$= \mathcal{N} \int_{C(x,y,t)} \exp\left[-\int_0^t L_{\mathrm{OM}}(X(u), \dot{X}(u))\, du\right] \prod_{0<s<t} dX(s) \tag{153}$$

where $e^{tA} \equiv T_t$ is the transition semigroup of the process considered, A the (forward) generator of this diffusion, $C(x, y, t)$ the set of continuous paths $X_u: [0, t] \to \mathbb{R}$ such that $X(0) = x$ and $X(t) = y$, and $\mathcal{N}$ a normalization. The function L_{OM} is called the Onsager–Machlup Lagrangian. In other words, since $(T_t f)(x)$ is

$$(T_t f)(x) = \int_{\mathbb{R}} f(y) p(t, y|0, x)\, dy \tag{154}$$

for any f bounded and measurable, if the right-hand side is defined, the problem is to represent the transition probability density $p(t, y|0, x)$ [defined by $p(X(t) \leq y | X(0) = x) = p(t, y|0, x)\, dy$] in terms of L_{OM} according to the formula (153).

In using a discretization procedure, the solution obtained (for a unity diffusion coefficient σ) was (Graham, 1978)

$$L_{\mathrm{OM}}(X, \dot{X}) = \tfrac{1}{2}[\dot{X} - b]^2 + \tfrac{1}{2} b_X \tag{155}$$

For example, for the standard Wiener process ($b = 0$, $\sigma = 1$) the potential sought is zero, only the "kinetic term" suvives.

Since only potentials invariant under coordinate transformations were supposed to have physical meaning, the whole program was presented for an n-dimensional diffusion process on a Riemannian manifold M.

From a more precise point of view, there exist at least two independent approaches to this type of problem, namely, an analytical approach and a probabilistic one. Let us illustrate the analytical approach with an example. Suppose that ψ_0 is the strictly positive ground state of the Hamiltonian operator on $L^2(\mathbb{R}, dx)$ with zero lowest eigenvalue

$$H = -\frac{1}{2}\frac{d^2}{dX^2} + V \tag{156}$$

It is well known that one can construct a diffusion process associated to H using the unitary equivalence between $L^2(\mathbb{R}, dX)$ and $L^2(\mathbb{R}, \psi_0^2(X)\, dX)$ under the unitary operator $U: g \mapsto \psi_0^{-1} g$. Then H is equivalent to (minus) the forward generator A of the diffusion sought, $-A = UHU^{-1}$, namely,

$$A = -g_X \frac{d}{dX} + \frac{1}{2}\frac{d^2}{dX^2} \tag{157}$$

for

$$g_X = -\frac{(\psi_0)_X}{\psi_0} \tag{158}$$

The right-hand term of (158) is noting more that the definition (107) of the forward drift b in stochastic mechanics.

Using $-A = UHU^{-1}$ and (158) one finds the relation between V and b,

$$V(X) = \tfrac{1}{2}[b^2 + b_X] \tag{159}$$

This is essentially the Onsager-Machlup potential of (155) associated to the stochastic differential equation (152) with unity diffusion coefficient σ. For a given potential V, equation (159) is actually a Riccati equation for the forward drift b and the definition (107) of b in term of ψ_0 enables us to linearize this Riccati equation in the time-independent Schrödinger equation for ψ_0. The operator

$$A = b\frac{d}{dX} + \frac{1}{2}\frac{d^2}{dX^2}$$

is in fact the generator of a hypercontractive semigroup on $L^2(\mathbb{R}, \psi_0^2(X)\,dX)$ and it is called a Dirichlet operator (Carmona, 1979b).

Observe that, since $T_t = e^{tA}$ on $L^2(\mathbb{R}, \psi_0^2(X)\,dX)$ is unitary equivalent to e^{-tH} on $L^2(\mathbb{R}, dX)$ the underlying diffusion process may also be interpreted as the result of an analytic continuation which replaces t by $-it$ in the propagator for the Schrödinger equation. Nelson (1984a) discussed at length in what sense this point of view is less natural than the real time stochastic mechanics interpretation.

The probabilistic approach of the Onsager-Machlup problem starts from a new statement of the initial question, namely, one looks for an asymptotic evaluation that the sample path $X(\cdot, \omega)$ belongs to a small tube of diameter 2ε around a chosen differentiable trajectory $x(t):[0, T]\to\mathbb{R}$ (more generally in M). If $p_{x(0)}$ denotes the probability measure for a process starting from $x(0)$ and whose forward generator is the operator A mentioned above, the solution is given by (Ikeda and Watanabe, 1981)

$$p_{x(0)}\left(\max_{s\in[0,t]}|X_s - x(s)| < \varepsilon\right)$$
$$\sim C\exp\left[-\frac{\lambda_1 t}{\varepsilon^2}\right]\exp\left[-\int_0^t L_{OM}(x(u), \dot{x}(u))\,du\right] \qquad \text{as } \varepsilon\downarrow 0 \tag{160}$$

where the first factor contains only constants and L_{OM} is (155). On a Riemannian manifold M (without boundary) with norm $\|\cdot\|$ the Onsager-Machlup Lagrangian $L_{OM}: TM\to\mathbb{R}$ takes the form (Takahashi and

Watanabe, 1981)

$$L_{\mathrm{OM}}(x, \dot{x}) = \tfrac{1}{2}\|\dot{x} - b(x)\|^2 + \tfrac{1}{2}\operatorname{div} b(x) - \tfrac{1}{12}R(X) \tag{161}$$

where R is the scalar curvature.

The presence of this strange curvature term in the Lagrangian is not a surprise for the physicists interested by the question of quantization on curved space. In particular, they know that the value of the constant factor in this term is a traditional object of dispute between the physicists (Schulman, 1981). Now let us examine the Onsager-Machlup problem from the point of view of stochastic calculus of variations.

5.2. Inverse Problem of Stochastic Calculus of Variations

In the deterministic case, the inverse problem of (classical) calculus of variations can be formulated in the following way. Given a second-order differential equation

$$\ddot{X} = f(X, \dot{X}, t) \tag{162}$$

for which Lagrangian $L_c = L_c(X, \dot{X}, t)$ does equation (162) coincide with the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{X}}\right) - \frac{\partial L_c}{\partial X} = 0? \tag{163}$$

As we know from classical mechanics, if such a Lagrangian exists, it is far to be unique. The general solution of this problem is difficult, but if the right-hand term of equation (162) is such that

$$f = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{X}}\right) - \frac{\partial U}{\partial X} \tag{164}$$

for some nice $U = U(X, \dot{X}, t)$ then

$$L_c(X, \dot{X}, t) = \tfrac{1}{2}|\dot{X}|^2 - U(X, \dot{X}, t) \tag{165}$$

is the simplest solution which corresponds to a Newtonian system with forces derivable from a potential.

By analogy with stochastic mechanics, the problem presented in Section 5.1 can be solved as follows. Given Itô's equation (152) with $X(0) = X_0(\omega)$ compute the "mean acceleration"

$$\tfrac{1}{2}(DD_*X + D_*DX) \tag{166}$$

with the help of the kinematical relations (21), (23), and (23'). If one can find a nice function $U = U(X, DX, D_*X, t)$ such that this result takes the form

$$D\frac{\partial U}{\partial D_* X} + D_* \frac{\partial U}{\partial DX} - \frac{\partial U}{\partial X} \tag{167}$$

then the Lagrangian

$$L(X, DX, D_*X, t) = \tfrac{1}{4}|DX|^2 + \tfrac{1}{4}|D_*X|^2 - U(X, DX, D_*X, t) \tag{168}$$

is a solution of this inverse problem of stochastic calculus of variations, namely, the process (152) is a stationary point of the action functional (24) for L. This point of view, which enables one to associate a "potential U" with a given stochastic process, may be called "thermal mechanics" in order to underline the distinction from stochastic mechanics (Zambrini, 1980b).

We illustrate the method by the simplest example, the one-dimensional standard Wiener process with initial measure concentrated at the origin [$b = 0$, $\sigma = 1$ in equation (152)]. By means of equations (23) and (23′), one obtains, for $X(t) = W(t)$,

$$\frac{1}{2}(DD_*X + D_*DX) = -\frac{X}{2t^2} \tag{169}$$

Then, one can choose here $U = U(X, t)$ in the definition (167) with

$$U = \frac{X^2}{4t^2} \tag{170}$$

Consequently, this Wiener process in $D_{X_0}^{X_T}$ is a stationary point of the action functional

$$J: X \mapsto E\left[\int_0^T \left\{\frac{1}{4}(DX)^2 + \frac{1}{4}(D_*X)^2 - \frac{X^2}{4t^2}\right\} dt\right] \tag{171}$$

The advantage of this indirect approach to the Onsager–Machlup problem lie in its formal relation to quantum mechanics. In the previous example, we know, thanks to stochastic mechanics, that equation (169) is associated to a Schrödinger equation for the potential $U = U(X, t)$ of (170). In particular, this relation enables us to define a natural "classical limit" for the given diffusion (here the Wiener process), namely, the one associated to the corresponding Schrödinger equation.

The relation between thermal mechanics and the classical Onsager–Machlup problem may seem superficial. This is not the case, however, as we shall see, if we start from a homogeneous diffusion process whose invariant measure is symmetrizable. Before that, we briefly recall the kinematical results necessary for defining stochastic mechanics on an n-dimensional Riemannian manifold M.

5.3. Stochastic Mechanics on a Riemannian Manifold

A generalization of Itô's stochastic differential equation (20) is given by

$$dX^i(t) = b^i(X, t)\, dt + e^i_j(t) \circ dW^j(t), \qquad X(0) = X_0 \tag{172}$$

$$de^i_j(t) = -\Gamma^i_{lk}(X) e^K_j(t) \circ dX^l(t) + \tfrac{1}{2} R^i_K(X) e^K_j(t)\, dt, \qquad e(0) = e_0 \tag{173}$$

Here $\{e^i_j(t)\}_{i,j=1 \text{ to } n}$ is an orthonormal base of the tangent space T_xM with

$$e^i_l e^j_l = C^{ij} \tag{174}$$

and $C = (C_{ij})$ denotes the Riemannian metric tensor. The $W^j(t)$ are n independent Wiener processes such that

$$E[dW^i(t)] = 0, \qquad E[dW^i(t)\, dW^j(t)] = \delta^{ij}\, dt \tag{175}$$

The first term in the right-hand side of equation (173) represents Itô's (1975b) generalization of the Levi-Cività parallel transport of the frame e along the curve $X(t)$. Indeed, let us recall that in the classical case, any vector $U^i(t)$ is transported parallelly along the smooth curve $X(t)$ in the sense of Levi-Cività if and only if

$$\delta U^i \equiv dU^i + \Gamma^i_{jK} U^j\, dX^K = 0, \qquad U(0) = U_0 \tag{176}$$

where the $\Gamma^i_{jk} = \Gamma^i_{jK}(X)$ are the Christoffel symbols. Since Itô's generalization uses the time symmetric Stratonovich integral $\circ$ in order to translate the orthonormal frame e from e_0 along the random curve $X(t)$, this operation is time symmetric. But it is not the convenient geometrical construction for stochastic mechanics. It is the origin of the supplementary term $\frac{1}{2} R^i_K e^K_j\, dt$ in (173), where R^i_K is the Ricci tensor, introduced by Dohrn and Guerra (1977, 1978). This operation is no longer reversible (for example the length of the transported vector is no longer conserved) but it enables us to define the suitable notion of stochastic acceleration.

A discussion of these questions may be found in Nelson (1984a) and Meyer (1982). The computation of a mean derivative like (6) becomes, for (172),

$$DX^i \equiv \lim_{\Delta t \downarrow 0} E\left[\frac{dX^i}{\Delta t}\middle| X(t)\right]$$

$$= b^i + \lim_{\Delta t \downarrow 0} E\left[\frac{e^i_j(t) \circ dW^j(t)}{\Delta t}\middle| X(t)\right]$$

Using the properties of the symmetric integral, equation (173), the rules of Itó's calculus, and the definitions (174) and (175) we get

$$DX^i \equiv \hat{b}^i = b^i - \tfrac{1}{2} \Gamma^i_{lK} C^{lK} \tag{177}$$

Now equation (173) is actually true for the stochastic parallel displacement of any vector U along X from $X(0)=X_0$ to $X(t)$; then one can introduce the linear operation of displacement

$$T_{X,t}\colon T_{X_0}M \to T_{X(t)}M$$
$$U(0) \mapsto U(t) \tag{178}$$

such that $U(t)$ is a solution of (173) (where U^i replaces e^i), and define the mean forward derivative of a vector field $f=f(X,t)$ by

$$Df(x(t),t) \equiv \lim_{\Delta t \downarrow 0} E\left[\frac{T^{-1}_{X,t+\Delta t}f(X(t+\Delta t), t+\Delta t) - f(X(t),t)}{\Delta t}\bigg| X(t)\right]$$

which yields

$$Df^i = \frac{\partial f^i}{\partial t} + \hat{b}^j \nabla_j f^i + \tfrac{1}{2}\Delta_{\mathrm{DR}} f^i \tag{179}$$

where Δ_{DR} is the Dohrn–Guerra operator such that

$$\Delta_{\mathrm{DR}} f^i = \nabla^j \nabla_j f^i + R^i_j f^j \tag{180}$$

The extra term $R^i_j f^j$ for the generator of the diffusion corresponds to the geodesic correction to Itô's (Levi-Cività) stochastic parallel displacement. Contrary to the Laplace–Beltrami operator, Δ_{DR} commutes with ∇, and it follows from this choice that the stochastic Euler–Lagrange equation for

$$L(X, DX, D_*X, t) = \frac{M}{4} DX^i DX_i + \frac{M}{4} D_*X^i D_*X_i - V(X,t) \tag{181}$$

leads to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^i \nabla_i \psi + V\psi \tag{182}$$

For further results concerning stochastic mechanics on a Riemannian manifold, consult Dankel (1971, 1977) and, for another use of this frame Nagasawa (1980). Now we come back to the Onsager–Machlup problem.

5.4. Onsager–Machlup Potential

We consider the inverse problem of stochastic calculus of variations for a diffusion process whose invariant measure $\rho_s(X)d_MX$ [where $d_MX = (\det C)^{1/2}\, dX^1 \cdots dX^n$] is symmetrizable on the Riemannian manifold M (Ikeda and Watanabe, 1981), namely, such that

$$\int_M T_t f(X) g(X) \rho_s(X)\, d_MX = \int_M f(X) T_t g(X) \rho_s(X)\, d_MX \tag{183}$$

for any smooth function $f, g: M \to \mathbb{R}$ in the domain of the transition semigroup T_t.

There exist numerous equivalent characterizations of this property. The most appropriate form for our purpose is the following relation between the (forward) drift b of the diffusion and the density ρ_s of its invariant measure (27)

$$b^i = \tfrac{1}{2}\nabla^i \log \rho_s \tag{184}$$

In physics, we generally call this equation the "detailed balance condition." If we use again the kinematical relations (96) and (101) introduced for stochastic mechanics (for $\hbar = M = 1$), we can easily verify that the detailed balance condition is equivalent to a vanishing current velocity

$$v^i \equiv \tfrac{1}{2}(b^i + b^i_*) = 0 \tag{185}$$

Now, it is a simple matter to find the form of the potential V in the stochastic Newton equation associated to the Lagrangian (181). We get

$$V = \tfrac{1}{2}[b^i b_i + \nabla^i b_i] \tag{186}$$

This is indeed the Onsager–Machlup Lagrangian we are looking for. The absence of the curvature term [compare with (161)] comes from the fact that, by construction, in stochastic mechanics, the classical limit of our Lagrangian (161) does not contain this factor whereas it must be included in the starting Lagrangian for the conventional quantization procedure (Schulman, 1981). Therefore, this solution of the inverse problem of stochastic calculus of variations gives us the complete dynamical information on the considered homogeneous diffusion process (Zambrini, 1980b).

By way of conclusion, we may observe that the stochastic variational frame presented in this review seems also adapted to problems in hydrodynamics (Nakagomi et al., 1981; Yasue, 1983).

NOTE ADDED IN PROOF

A retrospective look at the hypothesis used for the derivation of all these variational results suggests the following natural question: Is it really possible to *construct* diffusion processes satisfying these hypotheses and not only, as in this review, to assume their existence and to characterize their dynamics? The answer to this question is fortunately affirmative, and developed in a forthcoming publication (Zambrini, 1985).

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